Generating Random WMC Instances

An Empirical Analysis with Varying Primal Treewidth

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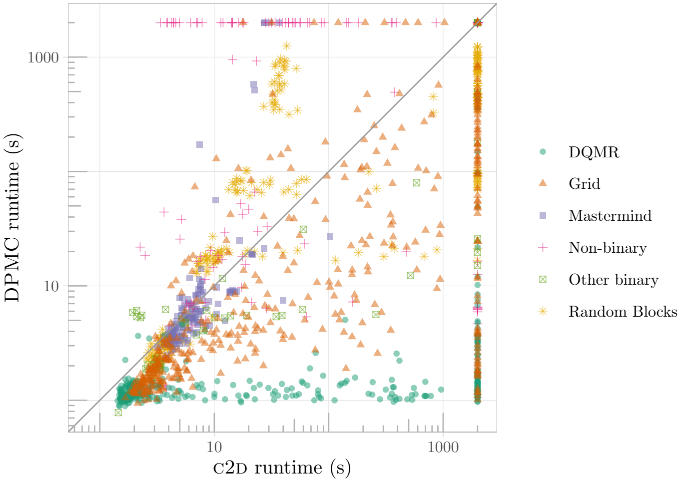
Which Algorithm Is Better? It Depends on the Data



The runtime data is from Dilkas and Belle (2021): various Bayesian networks encoded using the approach by Darwiche (2002)

The Problem: Weighted Model Counting (WMC)

- A generalisation of propositional model counting (#SAT)
- Applications:
  - graphical models
  - probabilistic programming
  - neuro-symbolic AI
- WMC algorithms use:
  - dynamic programming
  - knowledge compilation
  - SAT solvers

Example

$w(x) = 0.3,\ w(\neg x) = 0.7,$
$w(y) = 0.2,\ w(\neg y) = 0.8$

$\mathrm{WMC}(x \lor y) = w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

(Some of the) WMC Algorithms

- CACHET (Sang et al. 2004)
  - a SAT solver with clause learning and component caching
- C2D (Darwiche 2004)
  - knowledge compilation to d-DNNF
- D4 (Lagniez and Marquis 2017)
  - knowledge compilation to decision-DNNF
- MINIC2D (Oztok and Darwiche 2015)
  - knowledge compilation to decision sentential decision diagrams
- DPMC (Dudek, Phan and Vardi 2020)
  - dynamic programming with algebraic decision diagrams and tree decomposition based planning

Tree Decompositions and Primal Treewidth

Formula in CNF:

$$\phi = (x_4 \lor \neg x_3 \lor x_1) \land (\neg x_2 \lor x_4) \land (\neg x_1 \lor x_2 \lor x_4)$$

Tree Decompositions and Primal Treewidth

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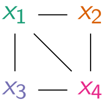
Its primal graph:



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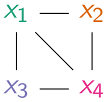
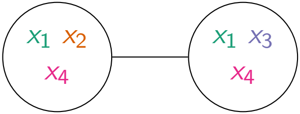


Its minimum-width tree decomposition:



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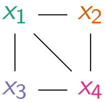

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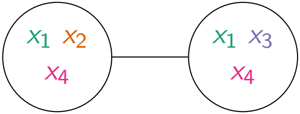


$\therefore$ the primal treewidth of ϕ is 2

The Parameterised Complexity of WMC Algorithms

Let n be the number of variables and m be the number of clauses.

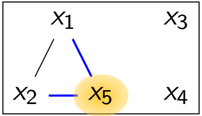
- Component caching (used in CACHET) is $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$, where w is the branchwidth of the underlying hypergraph (Bacchus, Dalmao and Pitassi 2009)
  - Branchwidth is within a constant factor of primal treewidth
- C2D is based on an algorithm, which is $\mathcal{O}(2^w mw)$, where w is at most primal treewidth (Darwiche 2001; Darwiche 2004)
- DPMC can be shown to be $\mathcal{O}(4^w mn)$, where w is an upper bound on primal treewidth

From Random SAT to Random WMC

We introduce parameter $\rho \in [0, 1]$ that biases the probability distribution towards adding variables that would introduce fewer new edges to the primal graph.

Example partially-filled formula:
$(\neg x_5 \lor x_2 \lor x_1) \land (x_5 \lor ? \lor ?)$

Its primal graph:



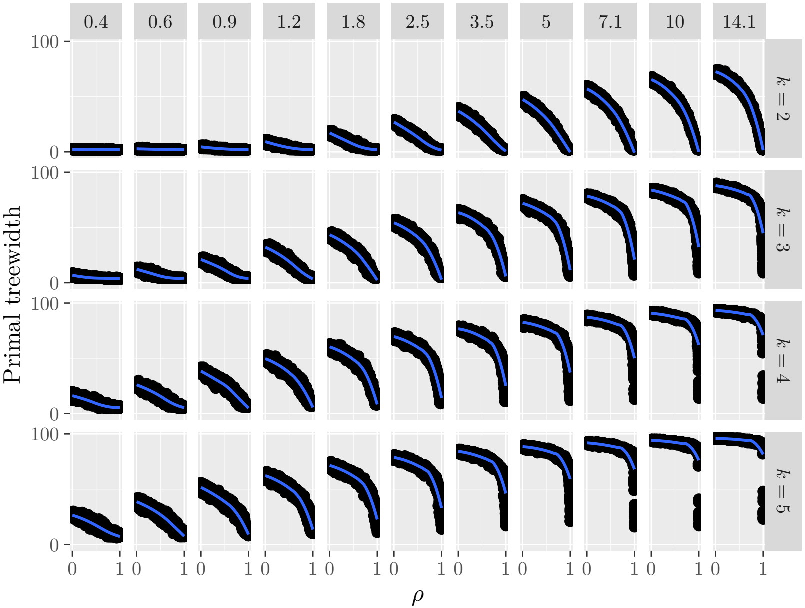
The probability distribution for the next variable

Base probability of each variable being chosen:

$$\frac{1-\rho}{4}.$$

Both x_1 and x_2 get a bonus probability of $\rho/2$ for each being the endpoint of one out of the two neighbourhood edges.

The Relationship Between ρ and Primal Treewidth



Peak Hardness w.r.t. Density

Let μ denote the density, i.e., the number of clauses divided by the number of variables.

- CACHET is known to peak at $\mu = 1.8$ (Sang et al. 2004)
- Bayardo Jr. and Pehoushek (2000) show some #SAT algorithms to peak at $\mu = 1.2$ and $\mu = 1.9$

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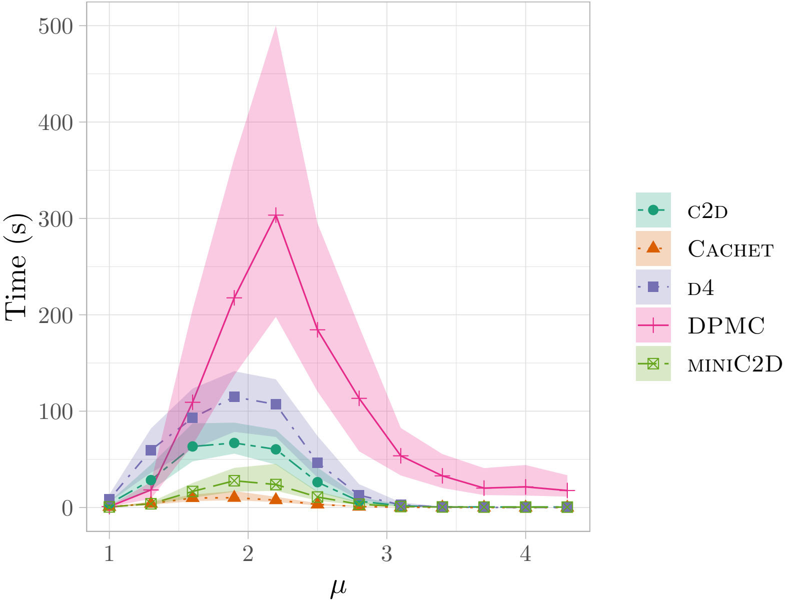
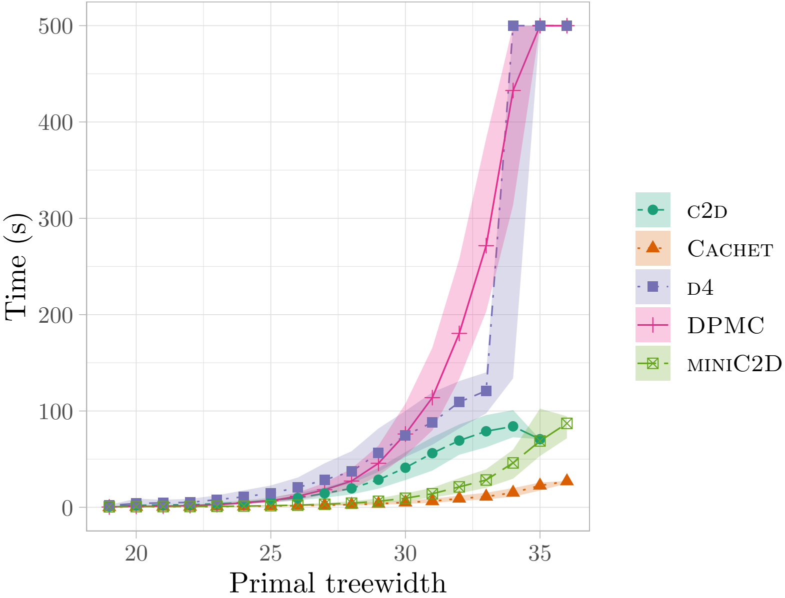
- In our experiments:
  - DPMC peaks at $\mu = 2.2$
  - all other algorithms peak at $\mu = 1.9$

Peak Hardness w.r.t. Density (when $\rho = 0$)



Hardness w.r.t. Primal Treewidth (when $\mu = 1.9$)



Is The Relationship Exponential?

Let us fit the model $\ln t \sim \alpha w + \beta$, i.e., $t \sim e^{\beta}(e^{\alpha})^{w}$, where t is runtime, and w is primal treewidth

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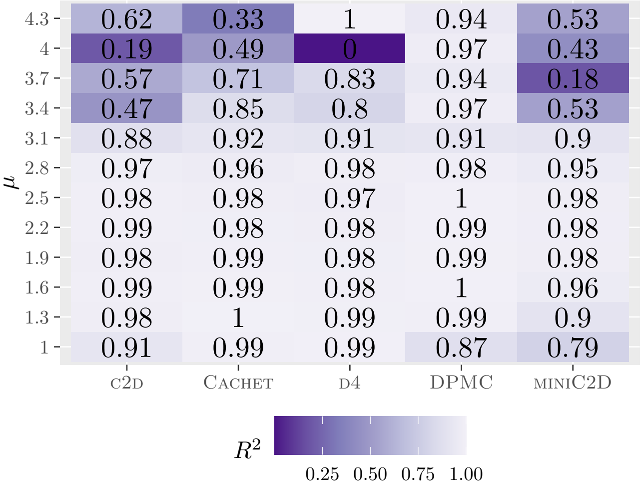

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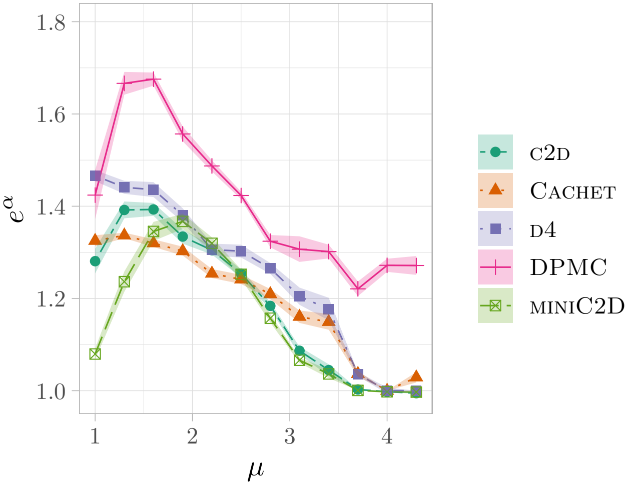
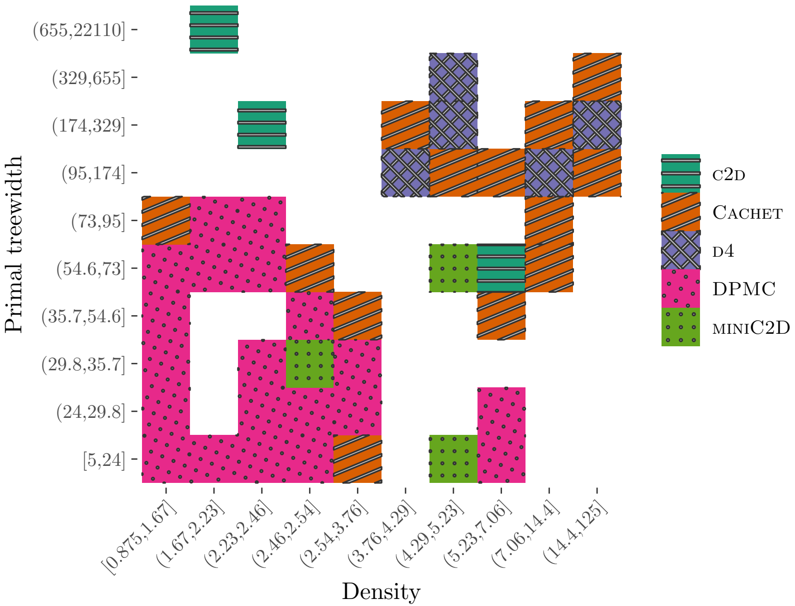
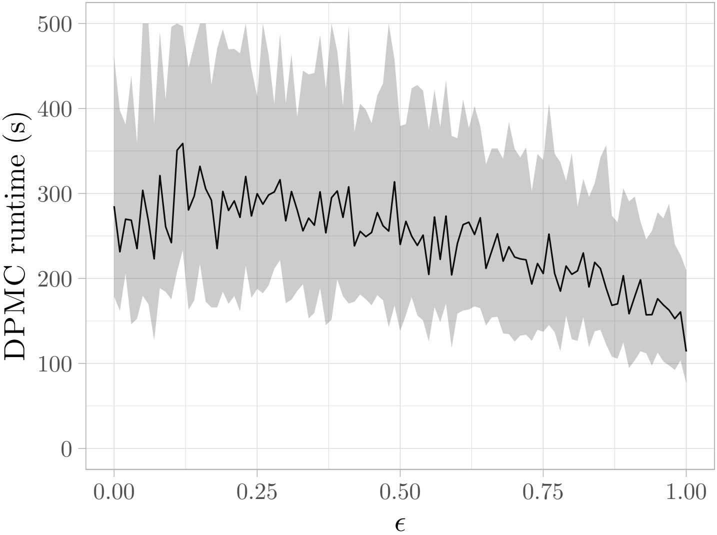


Does Real Data Confirm Our Observations?



Bonus: How DPMC Reacts to Redundancy in Weights

Let ϵ be the proportion of variables x s.t. $w(x) = w(\neg x) = 0.5$



Summary

- This work introduced a random model for WMC instances with a parameter that indirectly controls primal treewidth
- Observations:
  - All algorithms scale exponentially w.r.t. primal treewidth
  - The running time of DPMC:
    - peaks at a higher density
    - and scales worse w.r.t. primal treewidth
- Future work:
  - A theoretical relationship between ρ and primal treewidth
  - Non-k-CNF instances
  - Algorithm portfolios for WMC

Generating Random WMC Instances: The Algorithm

$\phi \leftarrow$ empty CNF formula;
$G \leftarrow$ empty graph;
for $i \leftarrow 1$ **to** m **do**
  $X \leftarrow \emptyset$;
  for $j \leftarrow 1$ **to** k **do**
    $x \leftarrow$ `newVariable(`X, G`)`;
    $\mathcal{V}(G) \leftarrow \mathcal{V}(G) \cup \{ x \}$;
    $\mathcal{E}(G) \leftarrow \mathcal{E}(G) \cup \{ \{ x, y \} \mid y \in X \}$;
    $X \leftarrow X \cup \{ x \}$;
  $\phi \leftarrow \phi \cup \{ \{ l \leftsquigarrow \mathcal{U}\{ x, \neg x \} \mid x \in X \} \}$;

- the number of clauses
- clause width
- a function to pick a variable
- a (fair) coin flip

How to Pick a Variable

Parameter $\rho \in [0, 1]$ biases the probability distribution towards adding variables that would introduce fewer new edges.

Function `newVariable`(*set of variables* X, *primal graph* G)**:**

$N \leftarrow \{\, e \in \mathcal{E}(G) \mid |e \cap X| = 1 \,\}$;

if $N = \emptyset$ **then return** $x \leftarrow\!\!\!\sim \mathcal{U}(\{\, x_1, x_2, \ldots, x_n \,\} \setminus X)$;

return

$$x \leftarrow\!\!\!\sim \left(\{\, x_1, x_2, \ldots, x_n \,\} \setminus X, y \mapsto \frac{1-\rho}{n-|X|} + \rho \frac{|\{\, z \in X \mid \{\, y, z \,\} \in \mathcal{E}(G) \,\}|}{|N|} \right);$$