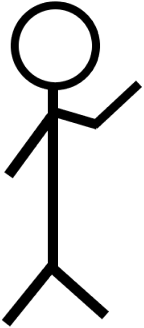
Synthesising Recursive Functions for First-Order Model Counting

Paulius Dilkas
Joint work with Vaishak Belle (Univeristy of Edinburgh, UK)

DTAI Seminar, 26th May 2023

National University of Singapore, Singapore

What Computers Can and Cannot Do



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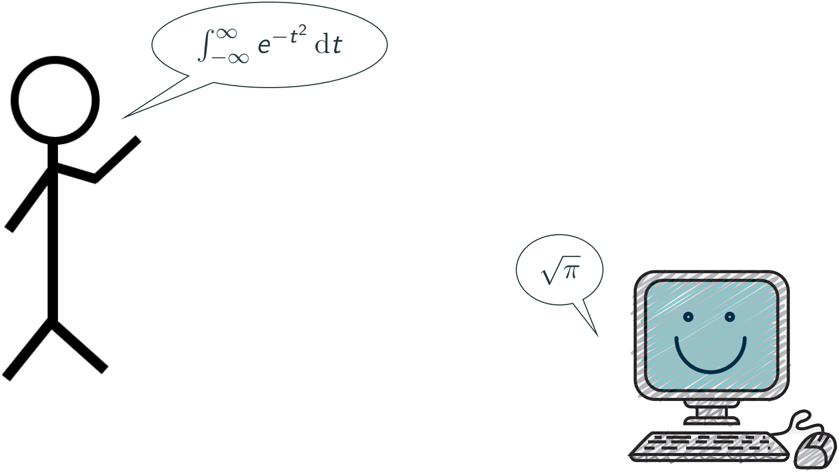

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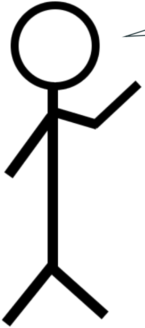
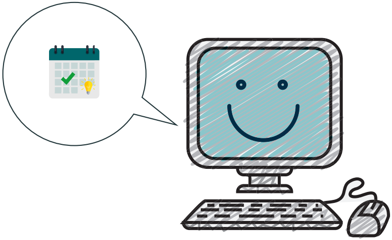


Produce a schedule for the nurses at the local hospital.



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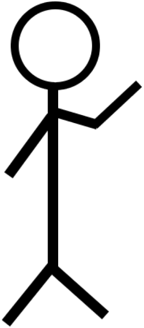
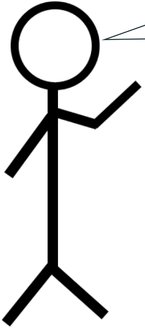
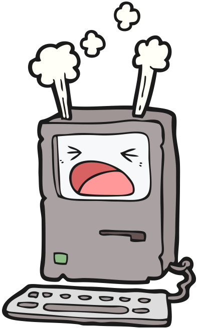


Paint a baroque oil painting of a raccoon queen wearing a crown.

What Computers Can and Cannot Do



If I shuffle a deck of n cards,
how many possible outcomes
are there?



Terms and conditions apply.

Who Cares About Counting?

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Probabilistic Programming

Inference and learning in probabilistic logic programs using weighted Boolean formulas

DAAN FIERENS, GUY VAN DEN BROECK, JORIS RENKENS, DIMITAR SHTERIONOV, BERND GUTMANN, INGO THON, GERDA JANSSENS and LUC DE RAEDT

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Neuro-symbolic AI

A Semantic Loss Function for Deep Learning with Symbolic Knowledge

Jingyi Xu [1] **Zilu Zhang** [2] **Tal Friedman** [1] **Yitao Liang** [1] **Guy Van den Broeck** [1]

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Natural Language Processing

Joint Inference for Knowledge Extraction from Biomedical Literature

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Robotics

Learning Relational Affordance Models for Robots in Multi-Object Manipulation Tasks

Bogdan Moldovan Plinio Moreno Martijn van Otterlo José Santos-Victor Luc De Raedt

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Bioinformatics

PheNetic: Network-based interpretation of unstructured gene lists in E. coli

Dries De Maeyer[1], Joris Renkens[2], Lore Cloots[1], Luc De Raedt[2], Kathleen Marchal[1,3]

[1]Center of Microbial and Plant Genetics, Kasteelpark Arenberg 20, B-3001, Leuven, Belgium
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Combinatorics

Automatic Conjecturing of P-Recursions Using Lifted Inference

Jáchym Barvínek[1(✉)], Timothy van Bremen[2], Yuyi Wang[3], Filip Železný[1], and Ondřej Kuželka[1]

[1] Czech Technical University in Prague, Prague, Czech Republic
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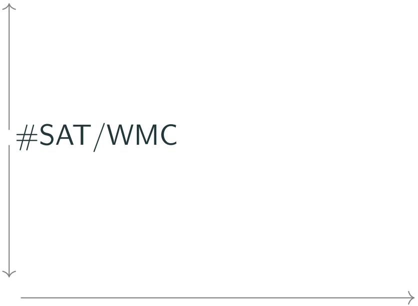
[2] KU Leuven, Leuven, Belgium
[3] ETH Zurich, Zurich, Switzerland

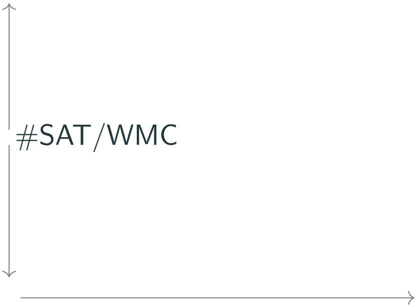
(Some of the) Many Ways to Count



#SAT (Valiant 1979)

- Input formula: $x \vee y$
- Interpretations: $\emptyset$, $\{x\}$, $\{y\}$, $\{x, y\}$
- Models: $\{x\}$, $\{y\}$, $\{x, y\}$

(Some of the) Many Ways to Count



#SAT (Valiant 1979)

- Input formula: $x \lor y$
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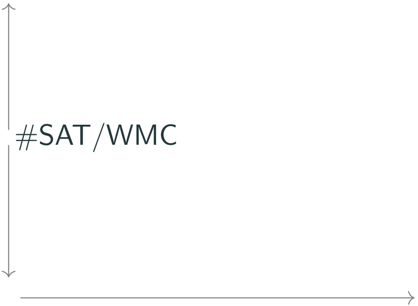
- Models: $\{x\}$, $\{y\}$, $\{x, y\}$
- Answer (model count): 3

(Some of the) Many Ways to Count

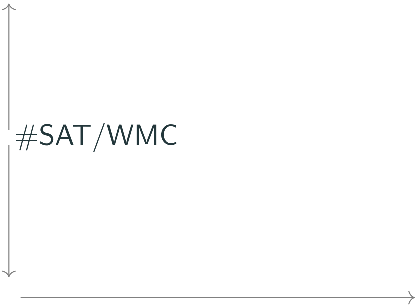


Weighted Model Counting **(Chavira and Darwiche 2008)**

- Input formula: $x \lor y$
- Input weights: $w(x) = 0.3$, $w(\neg x) = 0.7$,

  $w(y) = 0.2$, $w(\neg y) = 0.8$

(Some of the) Many Ways to Count



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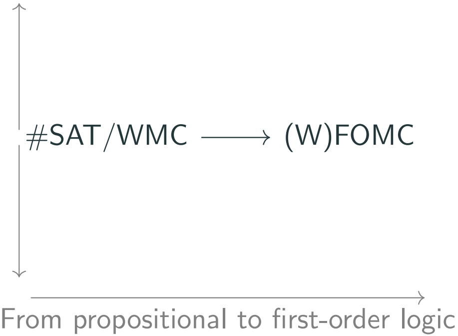
- Input formula: $x \vee y$
- Input weights: $w(x) = 0.3$, $w(\neg x) = 0.7$,

  $w(y) = 0.2$, $w(\neg y) = 0.8$
- Answer (weighted model count):

  $w(x)w(y) + w(x)w(\neg y) + w(\neg x)w(y) = 0.44$

(Some of the) Many Ways to Count



(Weighted) (Symmetric) First-Order Model Counting (Van den Broeck et al. 2011)

- Input formula: $\forall x \in \Delta.\ \mathtt{P}(x)$
- Input weights: $w^{+}(\mathtt{P}) = 0.3$, $w^{-}(\mathtt{P}) = 0.7$
- Input domain size(s): $|\Delta| = 2$

(Some of the) Many Ways to Count

#SAT/WMC ⟶ (W)FOMC

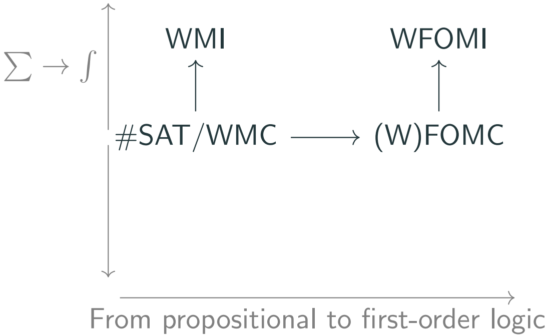
From propositional to first-order logic

(Weighted) (Symmetric) First-Order Model Counting

(Van den Broeck et al. 2011)

- Input formula: $\forall x \in \Delta.\ \texttt{P}(x)$
- Input weights: $w^{+}(\texttt{P}) = 0.3$, $w^{-}(\texttt{P}) = 0.7$
- Input domain size(s): $|\Delta| = 2$
- Answer: $(w^{+}(\texttt{P}))^{|\Delta|} = 0.09$

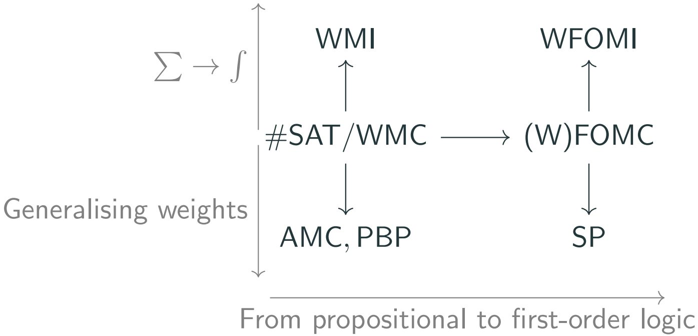
(Some of the) Many Ways to Count



Extensions to Continuous Domains

- Weighted model integration
  - (Belle, Passerini and Van den Broeck 2015)
- Weighted first-order model integration
  - (Feldstein and Belle 2021)

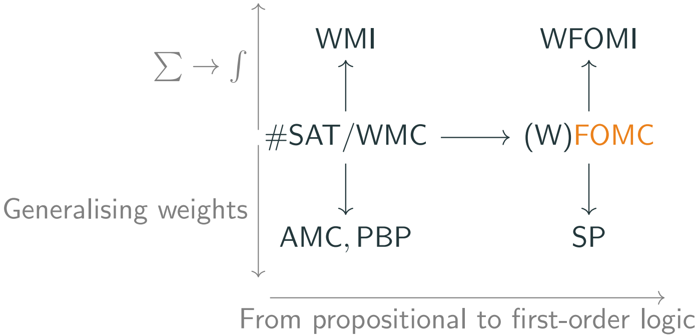
(Some of the) Many Ways to Count



Generalisations of the Weight Function

- Algebraic model counting
  - (Kimmig, Van den Broeck and De Raedt 2017)
  - From $\mathbb{R}_{\geq 0}$ to commutative semirings
- Pseudo-Boolean projection (D. and Belle 2021)
  - Weights not necessarily on literals
- Semiring programming (Belle and De Raedt 2020)

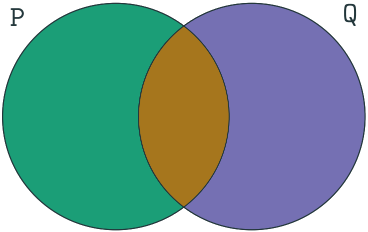
(Some of the) Many Ways to Count



(Unweighted) First-Order Model Counting

- Example formula:
  $\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$.
- Let $\Delta := \{ 1, 2 \}$.
- Interpretations: all subsets of
  $\{ \mathtt{P}(1), \mathtt{Q}(1), \mathtt{P}(2), \mathtt{Q}(2) \}$.

(Unweighted) First-Order Model Counting



- Example formula: $\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$.
- Let $\Delta \coloneqq \{1, 2\}$.
- Interpretations: all subsets of $\{\mathtt{P}(1), \mathtt{Q}(1), \mathtt{P}(2), \mathtt{Q}(2)\}$.
- Models:

$$\{\mathtt{P}(1), \mathtt{P}(2)\}, \quad \{\mathtt{P}(1), \mathtt{Q}(2)\}, \quad \{\mathtt{P}(1), \mathtt{P}(2), \mathtt{Q}(2)\},$$
$$\{\mathtt{Q}(1), \mathtt{P}(2)\}, \quad \{\mathtt{Q}(1), \mathtt{Q}(2)\}, \quad \{\mathtt{Q}(1), \mathtt{P}(2), \mathtt{Q}(2)\},$$
$$\{\mathtt{P}(1), \mathtt{Q}(1), \mathtt{P}(2)\}, \quad \{\mathtt{P}(1), \mathtt{Q}(1), \mathtt{Q}(2)\}, \quad \{\mathtt{P}(1), \mathtt{Q}(1), \mathtt{P}(2), \mathtt{Q}(2)\}.$$

Intuition

- Each 1-ary predicate is like a subset.
- For $n > 1$, each n-ary predicate is like a relation.
- FOMC counts combinations of relations.

More Formally: What Is the Input?

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ \mathtt{P}(x, y) \wedge \mathtt{P}(x, z) \Rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

More Formally: What Is the Input?

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- Any number of variables
- All variables are bound

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Many-Sorted Function-Free First-Order Logic with Equality

- Any number of variables
- All variables are bound
- $\forall$ and $\exists$ quantifiers can be nested arbitrarily deep
- All domains are finite
- Predicates can have any arity

Exact Algorithms for FOMC

- ForcLift (Van den Broeck et al. 2011)
    - knowledge compilation to FO d-DNNF
- L2C (Kazemi and Poole 2016)
    - knowledge compilation to C++ code
- Alchemy (Gogate and Domingos 2016)
    - DPLL-style search
- FastWFOMC (van Bremen and Kuželka 2021)
    - knowledge compilation to sd-DNNF

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Our Contribution

ForcLift

$+$

Recursion

$=$

Crane

ForcLift and First-Order Knowledge Compilation

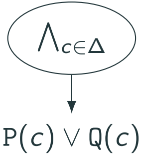
$$\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$$

ForcLift and First-Order Knowledge Compilation

$$\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$$

Independent partial grounding (introduces a constant $c \in \Delta$)

ForcLift and First-Order Knowledge Compilation



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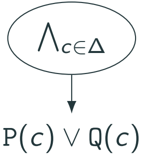

Shannon decomposition (a.k.a. Boole's expansion theorem) on $\mathtt{Q}(c)$

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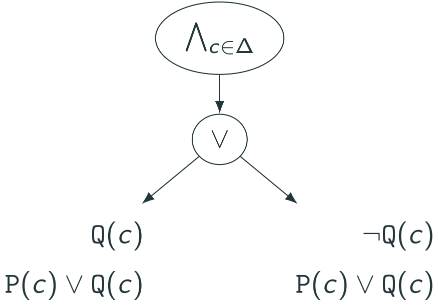

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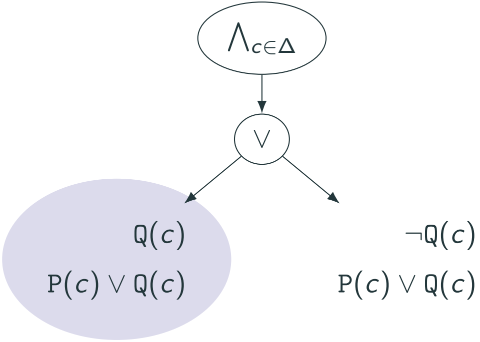

Positive unit propagation of $\mathtt{Q}(c)$

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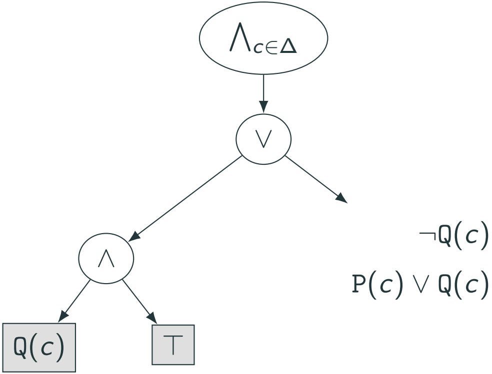

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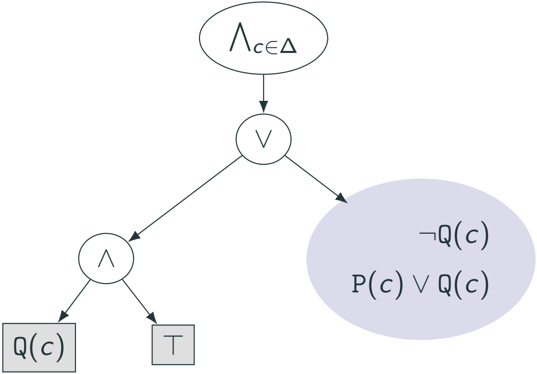

Negative unit propagation of $\neg\mathtt{Q}(c)$

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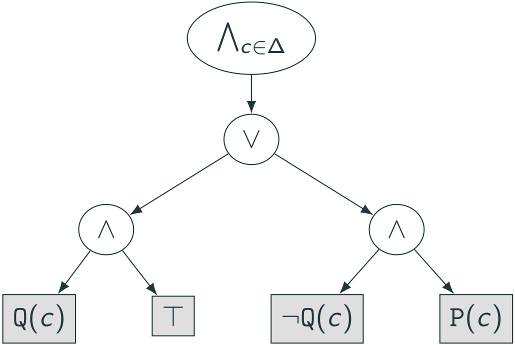

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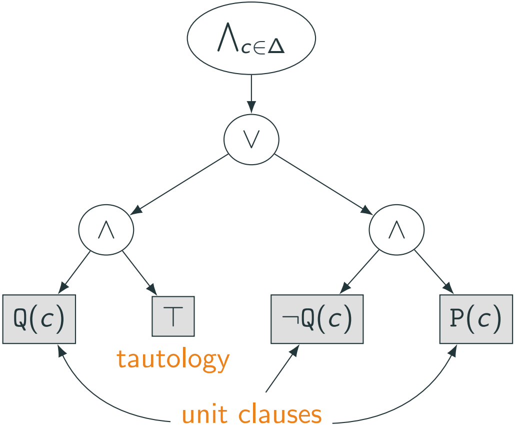
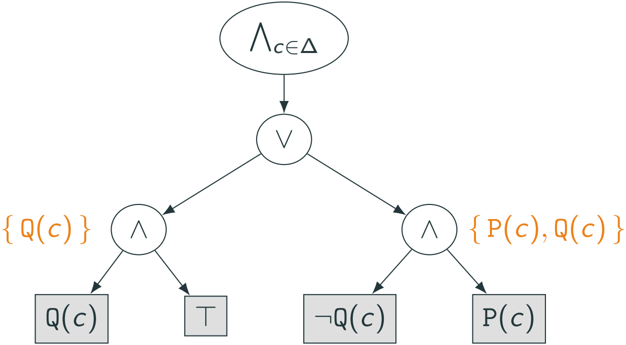


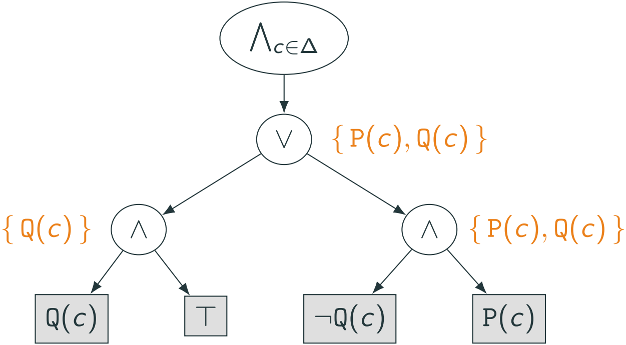
Compilation is complete ✓

ForcLift and First-Order Knowledge Compilation



Smoothing: propagating atoms upwards

ForcLift and First-Order Knowledge Compilation



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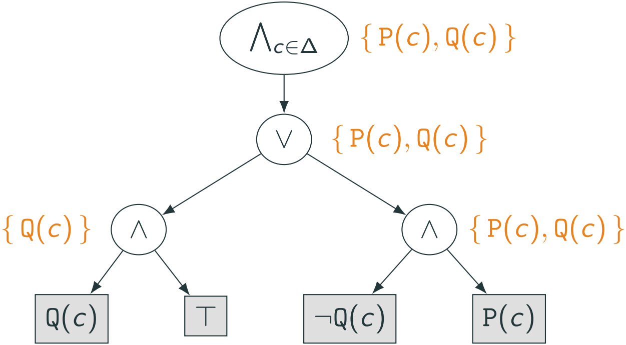

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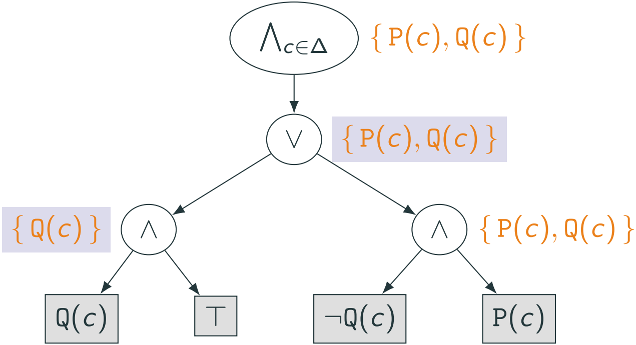

Smoothing: adding new atoms

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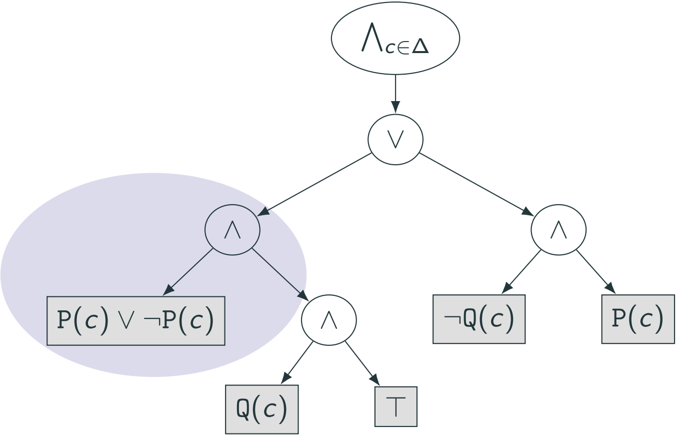

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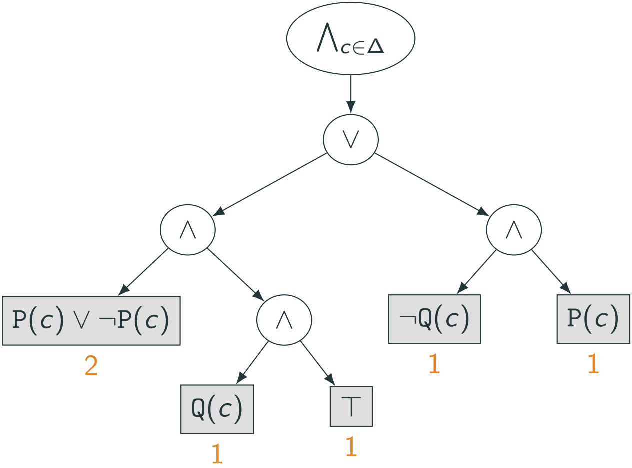

Propagating the model count

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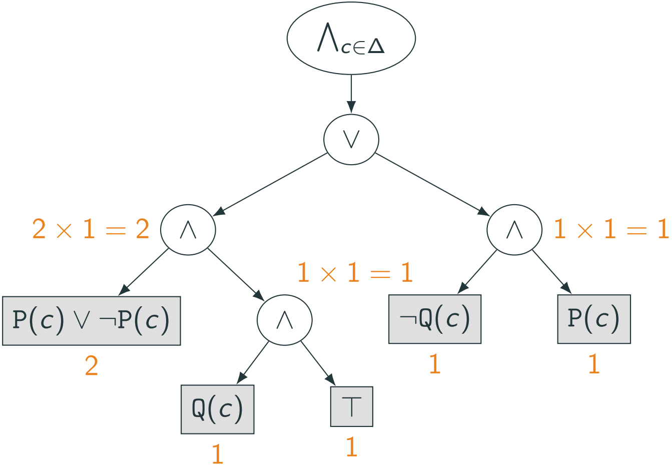
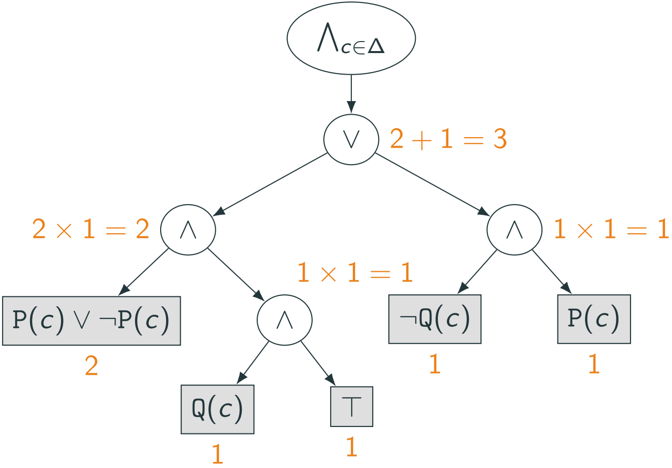


Propagating the model count

ForcLift and First-Order Knowledge Compilation



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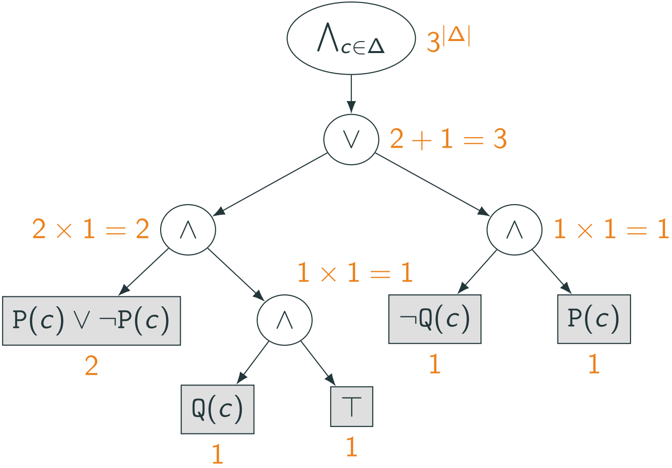

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A (Slightly) More Complicated Example

Suppose this room has n seats, and there are $m \leq n$ people in the audience. How many ways are there to seat everyone?

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More explicitly, we assume that:

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- and a seat can accommodate at most one person.

Answer: $n^{\underline{m}} = n \cdot (n-1) \cdots (n-m+1)$.

Note: this problem is equivalent to counting $[m] \to [n]$ injections.

Let's Express This Problem in Logic!

- Let Γ and Δ be sets (i.e., domains)
  - such that $|\Gamma| = m$, and $|\Delta| = n$.
- Let $\mathtt{P} \subseteq \Gamma \times \Delta$ be a relation (i.e., predicate) over Γ and Δ.
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$$\forall x \in \Gamma.\ \exists y \in \Delta.\ \mathtt{P}(x, y) \tag{1}$$

  - one person cannot occupy multiple seats

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ \mathtt{P}(x, y) \land \mathtt{P}(x, z) \Rightarrow y = z \tag{2}$$

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(1) and (2) constrain $\mathtt{P}$ to be a function, and (3) makes it injective.

Recursion

Back to Our Example

The following function counts injections:

$$f(m, n) = \begin{cases} 1 & \text{if } m = 0 \text{ and } n = 0 \\ 0 & \text{if } m > 0 \text{ and } n = 0 \\ f(m, n-1) + m \times f(m-1, n-1) & \text{otherwise.} \end{cases}$$

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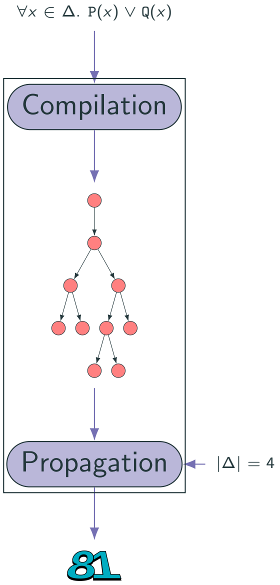
- Optimal time complexity to compute $n^{\underline{m}}$ is $\Theta(m)$.
- But $\Theta(mn)$ is still much better than translating to propositional logic and solving a #P-complete problem.
- The rest of this talk is about how to construct such functions automatically.

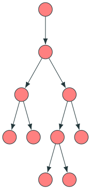
First-Order Knowledge Compilation: Before and After



First-Order Knowledge Compilation: Before and After

$\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$

Compilation



Propagation ← $|\Delta| = 4$

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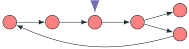
$\forall x \in \Gamma.\ \exists y \in \Delta.\ \mathtt{P}(x, y)$

$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ \mathtt{P}(x, y) \land \mathtt{P}(x, z) \Rightarrow y = z$

$\forall w, x \in \Gamma.\ \forall y \in \Delta.\ \mathtt{P}(w, y) \land \mathtt{P}(x, y) \Rightarrow w = x$

Compilation



Conversion

$f(m, n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m - l, n - 1)$

Simplification

$f(m, n) = f(m, n - 1) + m \times f(m - 1, n - 1)$

Evaluation ← $f(0, 0) = 1,\ f(m, 0) = 0$, $m = 2,\ n = 7$

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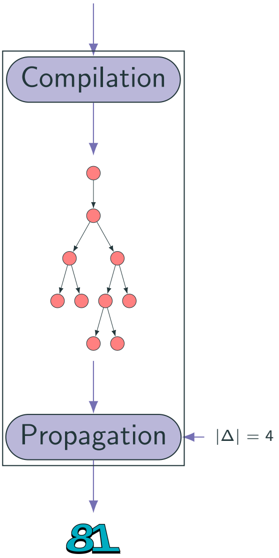
First-Order Knowledge Compilation: Before and After

$\forall x \in \Delta.\ \mathtt{P}(x) \lor \mathtt{Q}(x)$



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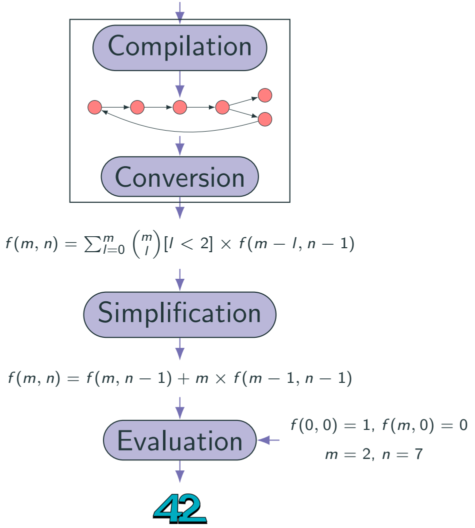

Circuits vs Graphs

Circuits (Van den Broeck et al. 2011)...

- ... extend d-DNNF circuits (Darwiche 2001) for propositional knowledge compilation with more node types
- ... are acyclic.

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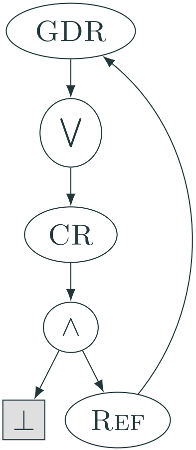
- ...are acyclic.

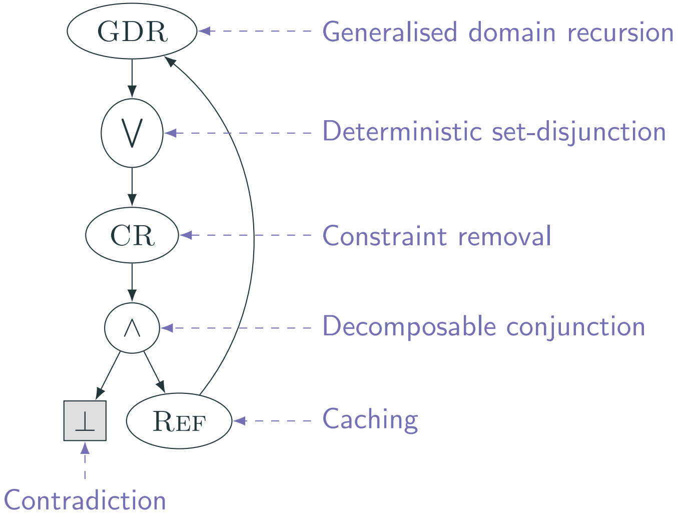
First-Order Computational Graphs (FCGs) are...
directed ~~acyclic~~ (weakly connected) graphs with:

- a single source,
- labelled nodes,
- and ordered outgoing edges.

How to Interpret an FCG



How to Interpret an FCG



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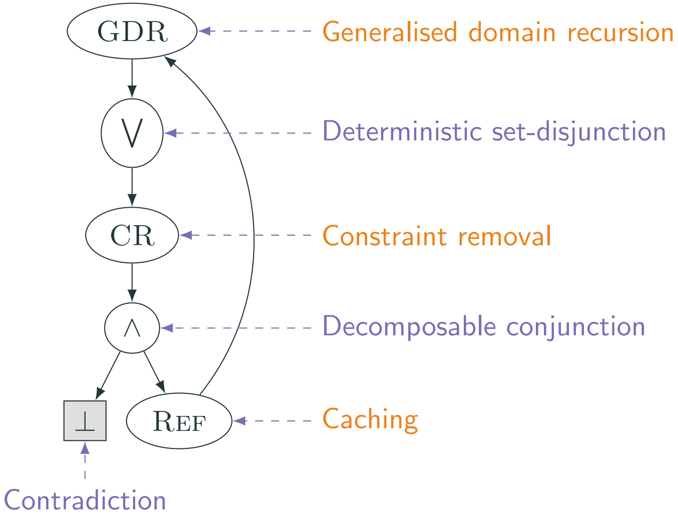

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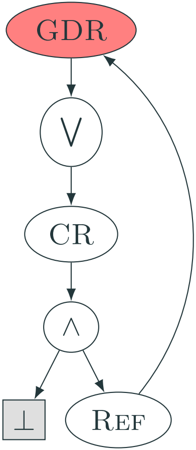
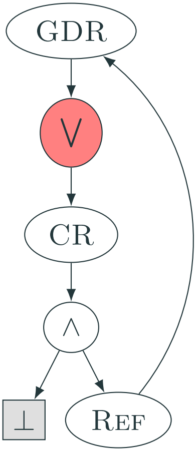


$f(m, n) =$

How to Interpret an FCG



$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l}$$

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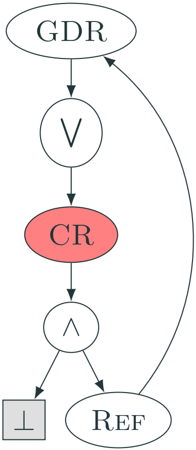

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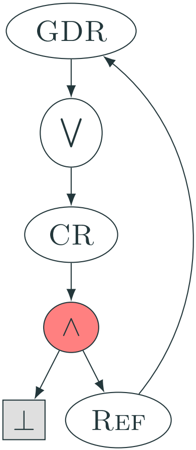

$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} \qquad \times$$

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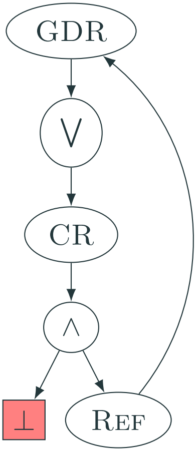
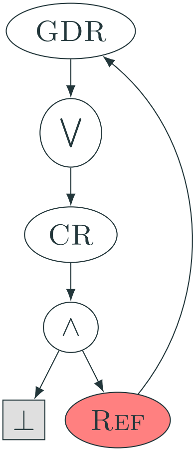


$$f(m, n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times$$

$$[\phi] = \begin{cases} 1 & \text{if } \phi \\ 0 & \text{if } \neg\phi \end{cases}$$

How to Interpret an FCG



$$f(m,n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m-l, n-1)$$

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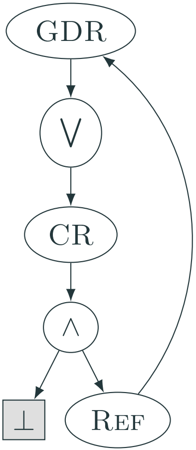
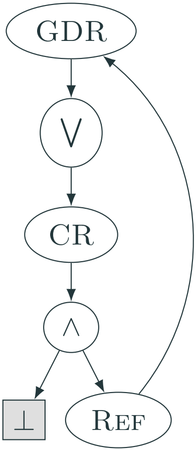


$$f(m, n) = \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m - l, n - 1)$$
$$= \binom{m}{0} \times f(m - 0, n - 1)$$
$$+ \binom{m}{1} \times f(m - 1, n - 1)$$

How to Interpret an FCG



$$
\begin{aligned}
f(m,n) &= \sum_{l=0}^{m} \binom{m}{l} [l < 2] \times f(m-l, n-1) \\
&= \binom{m}{0} \times f(m-0, n-1) \\
&+ \binom{m}{1} \times f(m-1, n-1) \\
&= f(m, n-1) + m \times f(m-1, n-1)
\end{aligned}
$$

Compilation: How FCGs Are Built

Definition
A (compilation) rule is a function that takes a formula and returns a set of (G, L) pairs, where

- G is a (possibly incomplete) FCG,
- and L is a list of formulas.

The formulas in L are then compiled, and the resulting FCGs are inserted into G according to a set order.

Example Compilation Rule: Independence

Input formula:

$$(\forall x, y \in \Omega.\ x = y) \land \tag{1}$$

$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ \mathtt{P}(x, y) \land \mathtt{P}(x, z) \Rightarrow y = z) \land \tag{2}$$

$$(\forall w, x \in \Gamma.\ \forall y \in \Delta.\ \mathtt{P}(w, y) \land \mathtt{P}(x, y) \Rightarrow w = x) \tag{3}$$

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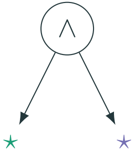
$$(\forall x, y \in \Omega.\ x = y) \wedge \quad (1)$$

$$(\forall x \in \Gamma.\ \forall y, z \in \Delta.\ \mathtt{P}(x, y) \wedge \mathtt{P}(x, z) \Rightarrow y = z) \wedge \quad (2)$$

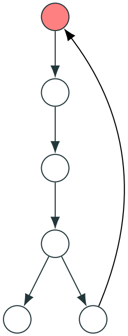
$$(\forall w, x \in \Gamma.\ \forall y \in \Delta.\ \mathtt{P}(w, y) \wedge \mathtt{P}(x, y) \Rightarrow w = x) \quad (3)$$

The independence compilation rule returns one (G, L) pair:

$G =$ 

$, \qquad L = \langle (1), (2) \wedge (3) \rangle$

New Rule 1/3: Generalised Domain Recursion



Example

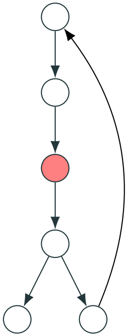
Input formula:

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$

Output formula (with a new constant $c \in \Gamma$):

$$\forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(c, y) \lor \neg \mathrm{P}(c, z)$$

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ x \neq c \land y \neq z \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$



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$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ x \neq c \land y \neq z \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$

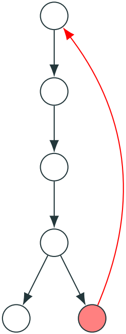
$$\forall w, x \in \Gamma.\ \forall y \in \Delta.\ w \neq c \land x \neq c \land w \neq x \Rightarrow \neg \mathrm{P}(w, y) \lor \neg \mathrm{P}(x, y)$$

Output formula (with a new domain $\Gamma' := \Gamma \setminus \{ c \}$):

$$\forall x \in \Gamma'.\ \forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$

$$\forall w, x \in \Gamma'.\ \forall y \in \Delta.\ w \neq x \Rightarrow \neg \mathrm{P}(w, y) \lor \neg \mathrm{P}(x, y)$$

New Rule 3/3: Identifying Possibilities for Recursion



Goal
Check if the input formula is equivalent (up to domains) to a previously encountered formula.

Outline

1. Consider pairs of 'similar' clauses.
2. Consider bijections between their sets of variables.
3. Extend each such bijection to a map between sets of domains.
4. If the bijection makes the clauses equivalent, and the domain map is compatible with previous domain maps, move on to another pair of clauses.

How These Rules Fit Together (1/5)

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$

$$\forall w, x \in \Gamma.\ \forall y \in \Delta.\ w \neq x \Rightarrow \neg \mathrm{P}(w, y) \lor \neg \mathrm{P}(x, y)$$

Generalised domain recursion

$$\forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(c, y) \lor \neg \mathrm{P}(c, z)$$

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ y \neq z \land x \neq c \Rightarrow \neg \mathrm{P}(x, y) \lor \neg \mathrm{P}(x, z)$$

$$\forall x \in \Gamma.\ \forall y \in \Delta.\ x \neq c \Rightarrow \neg \mathrm{P}(c, y) \lor \neg \mathrm{P}(x, y)$$

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How These Rules Fit Together (2/5)

$$\forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(c, y) \vee \neg \mathrm{P}(c, z)$$

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ y \neq z \wedge x \neq c \Rightarrow \neg \mathrm{P}(x, y) \vee \neg \mathrm{P}(x, z)$$

$$\forall x \in \Gamma.\ \forall y \in \Delta.\ x \neq c \Rightarrow \neg \mathrm{P}(c, y) \vee \neg \mathrm{P}(x, y)$$

$$\forall w \in \Gamma.\ \forall y \in \Delta.\ w \neq c \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(c, y)$$

$$\forall w, x \in \Gamma.\ \forall y \in \Delta.\ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(x, y)$$

↓

Atom counting and unit propagation

↓

$$\forall y, z \in \Delta^{\top}.\ y \neq z \Rightarrow \bot$$

$$\forall x \in \Gamma.\ \forall y, z \in \Delta^{\bot}.\ y \neq z \wedge x \neq c \Rightarrow \neg \mathrm{P}(x, y) \vee \neg \mathrm{P}(x, z)$$

$$\forall w, x \in \Gamma.\ \forall y \in \Delta^{\bot}.\ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(x, y)$$

How These Rules Fit Together (3/5)

$$\forall y, z \in \Delta^{\top}.\ y \neq z \Rightarrow \bot$$

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$$\forall w, x \in \Gamma.\ \forall y \in \Delta^{\bot}.\ w \neq x \wedge w \neq c \wedge x \neq c \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(x, y)$$

Constraint removal

$$\forall y, z \in \Delta^{\top}.\ y \neq z \Rightarrow \bot$$

$$\forall x \in \Gamma'.\ \forall y, z \in \Delta^{\bot}.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \vee \neg \mathrm{P}(x, z)$$

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How These Rules Fit Together (4/5)

$$\forall y, z \in \Delta^{\top}.\ y \neq z \Rightarrow \bot$$

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$$\forall w, x \in \Gamma'.\ \forall y \in \Delta^{\bot}.\ w \neq x \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(x, y)$$

Independence

$$\forall y, z \in \Delta^{\top}.\ y \neq z \Rightarrow \bot$$

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How These Rules Fit Together (5/5): Recursion

$$\forall x \in \Gamma'.\ \forall y, z \in \Delta^{\perp}.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \vee \neg \mathrm{P}(x, z)$$
$$\forall w, x \in \Gamma'.\ \forall y \in \Delta^{\perp}.\ w \neq x \Rightarrow \neg \mathrm{P}(w, y) \vee \neg \mathrm{P}(x, y)$$

$$\|$$

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ y \neq z \Rightarrow \neg \mathrm{P}(x, y) \vee \neg \mathrm{P}(x, z)$$
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$$+$$

$$\{ \Gamma \mapsto \Gamma', \Delta \mapsto \Delta^{\perp} \}$$

Resulting Improvements to Counting Functions

Let Γ and Δ be two sets with cardinalities $|\Gamma| = m$ and $|\Delta| = n$.
Our new rules enable Crane to efficiently count $\Gamma \to \Delta$ functions such as:

- injections in $\Theta(mn)$ time
  - by hand: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
  - by hand: $\Theta(\min\{ m, n \}^2)$
- bijections in $\Theta(m)$ time
  - optimal!

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Our new rules enable Crane to efficiently count $\Gamma \to \Delta$ functions such as:

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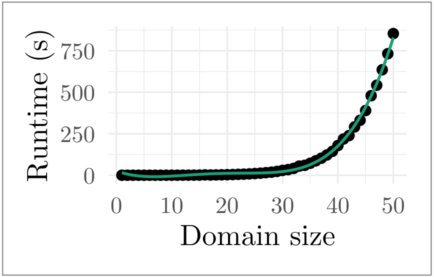
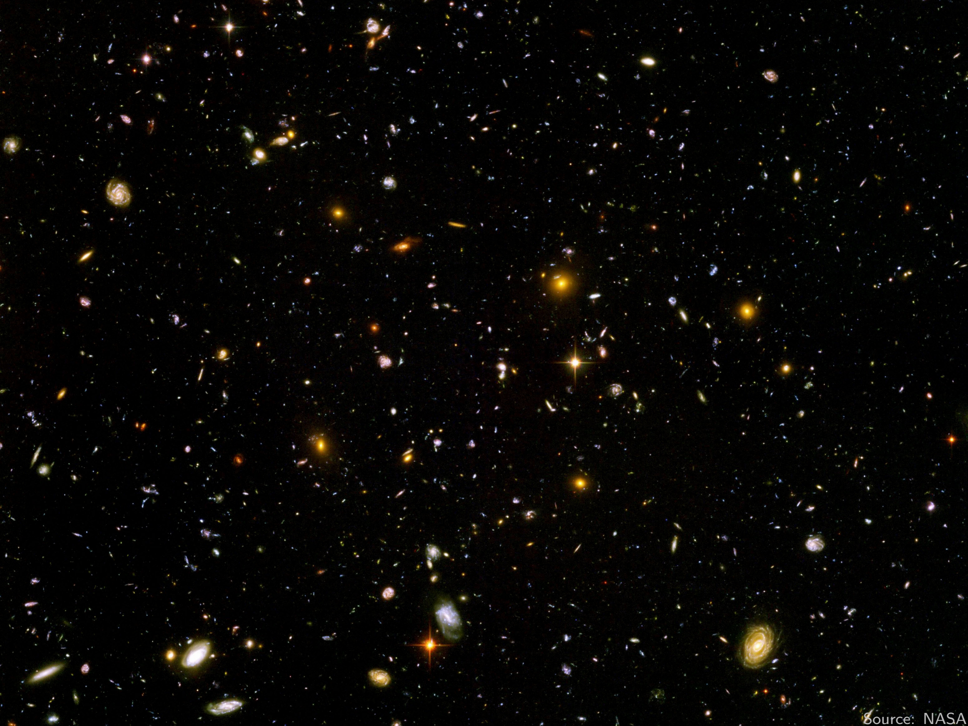
  - by hand: $\Theta(m)$
- partial injections in $\Theta(mn)$ time
  - by hand: $\Theta(\min\{ m, n \}^2)$
- bijections in $\Theta(m)$ time
  - optimal!
  - In comparison, FastWFOMC scales as $\Omega(m^4)$.





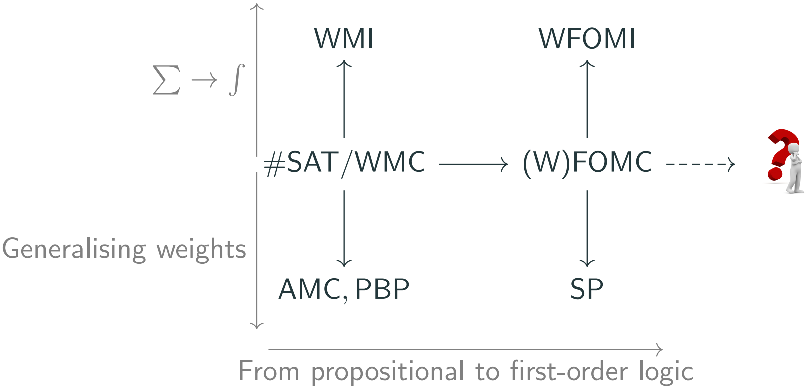
Source: NASA

What Have We Learned?

- Knowledge compilation can build graphs with cycles.
- Graphs (as well as circuits) define functions.
- Cycles can represent recursive calls, including:
  - mutual recursion
  - and function calls as complex as $f(n-k-2)$.
- Recursion helps us solve counting problems that were previously beyond the reach of FOMC.
- In some cases, even if a polynomial-time solution is already known, Crane is able to find more efficient solutions, with a lower degree polynomial.


Future →
← Past

Beyond First-Order Logic



What kind of logic is needed to succinctly describe, e.g.,

- $f(n) = f(f(n-1))$
- or the Fibonacci sequence?

Algebraic Solutions to Parameterised Problems (1/2)

- Suppose we have a Markov logic network that models the probability P that some system will fail.
- Here:
  - domain sizes describe the numbers of various components,
  - and weights express probabilities that:
    - some component fails,
    - or some combination of failures leads to another failure.
- Crane can express P as a function of the domain sizes and weights.

Algebraic Solutions to Parameterised Problems (2/2)

With the help of a computer algebra system, we can then:

- determine how P scales with the number of users,
- find combinations of domain sizes that keep P below some threshold,
- find ranges of weights that keep P sufficiently small across a range of domain size values.

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Reasoning with functions $>$ Reasoning with numbers