Towards Practical First-Order Model Counting

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SAT 2025

Motivation

Example Setting

- Let Δ be a set of cardinality $n \in \mathbb{N}_0$
- Suppose we want to count all $P \subseteq \Delta^2$ (as a function of n) that are:
  - functions,
  - bijections,
  - partial orders,
  - symmetric,
  - transitive,
  - etc.

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👎 Propositional model counting (#SAT) is #P-complete

👍 But many of these counting problems have efficient solutions

- And we can find them using first-order model counting
  - i.e., reasoning about sets, subsets, and arbitrary elements without grounding them

More Formally: What Is the Input?

Example Input Sentence

$$\forall x \in \Gamma.\ \forall y, z \in \Delta.\ P(x, y) \wedge P(x, z) \rightarrow y = z$$

Many-Sorted Function-Free First-Order Logic with Equality

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- $\exists$ and $\forall$ quantifiers can be nested arbitrarily deeply
- All domains are finite
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First-Order Model Counting (FOMC)

- Each predicate acts like a subset
  - of a domain or a Cartesian product of domains
- Goal: count combinations of subsets that satisfy the sentence

Exact Algorithms for FOMC

Predecessors of This Work

- ForcLift (Van den Broeck et al. 2011)
  - knowledge compilation to FO d-DNNF
- Crane (Dilkas and Belle 2023)
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  - extends ForcLift with support for:
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Some Other Approaches

- L2C (Kazemi and Poole 2016)
  - knowledge compilation to C++ code
- Alchemy (Gogate and Domingos 2016)
  - DPLL-style search
- FastWFOMC (van Bremen and Kuželka 2021)
  - based on cell enumeration

Previous Work: Crane (Dilkas and Belle 2023)

- A knowledge compilation approach:
  - Sentences $\rightarrow$ labelled digraphs $\rightarrow$ function-defining equations
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An Example Solution for Counting Bijections

$$f(m,n) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} g(l,m),$$

$$g(l,m) = g(l-1,m) + mg(l-1,m-1)$$

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Issues We Are Going to Address

Completeness: recursive functions (like g) have no base cases

Usability: how do I compute, e.g., $f(7,7)$? (C++ to the rescue!)

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3. ($\Rightarrow$) Identify a sufficient set of base cases of all recursive functions
   - e.g., $\{ g(0,m), g(l,0) \}$

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4.1. ($\Rightarrow$) Construct the corr. sentence

$\forall y \in \Delta.\ S(y) \vee \neg S(y)$ $\qquad$ $\top$

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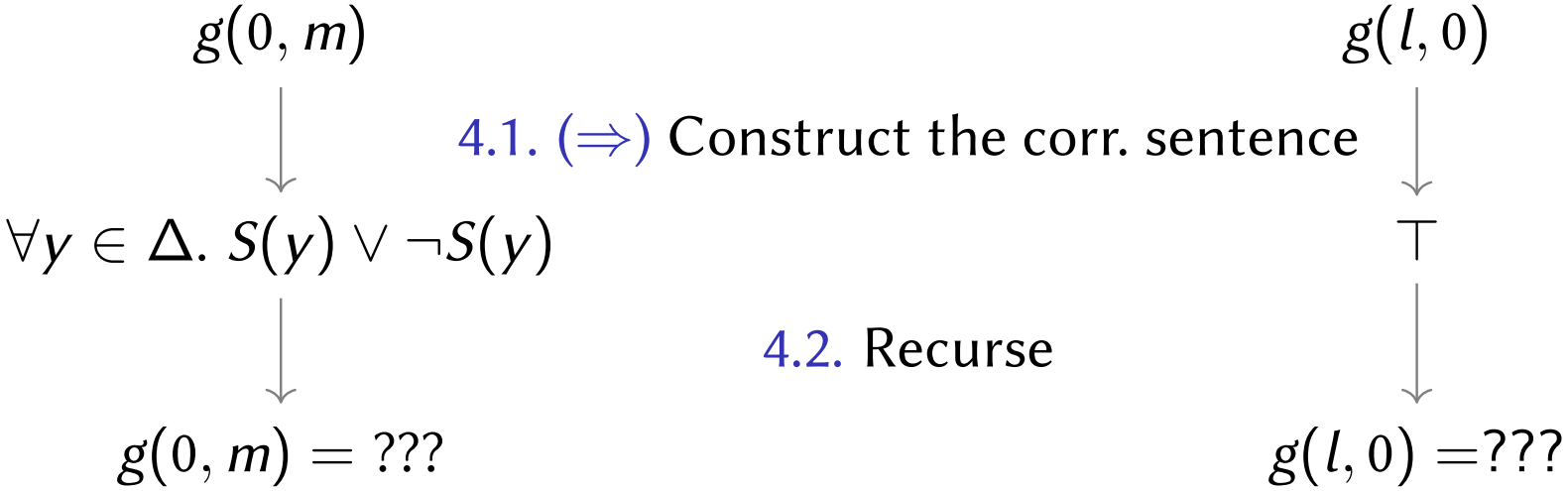

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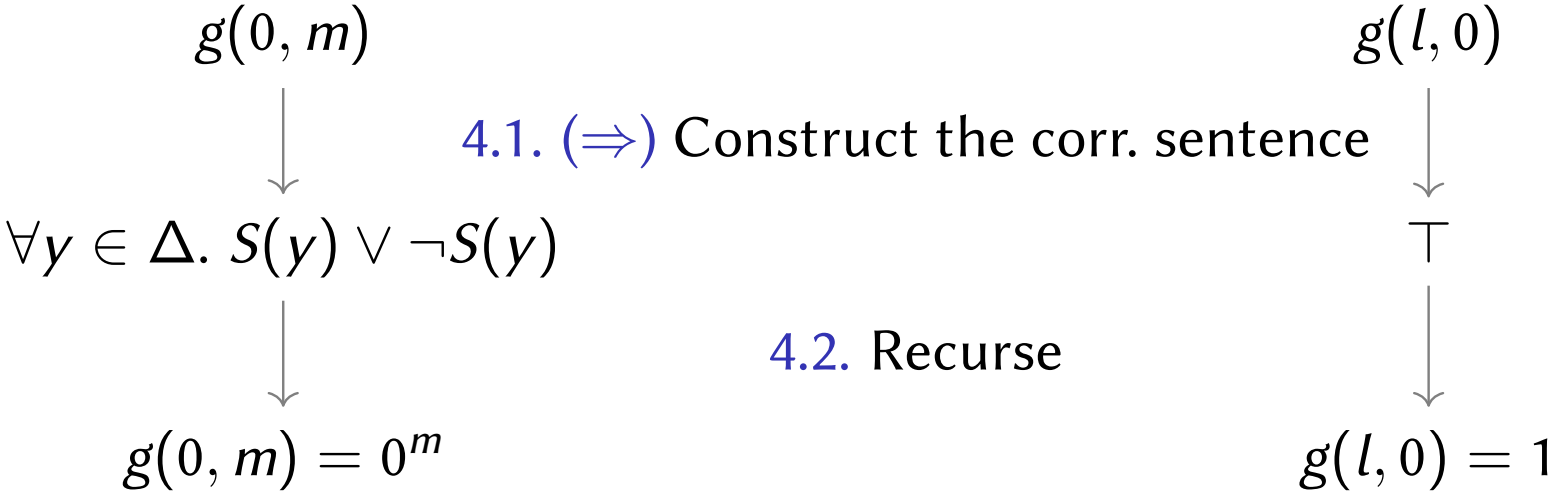

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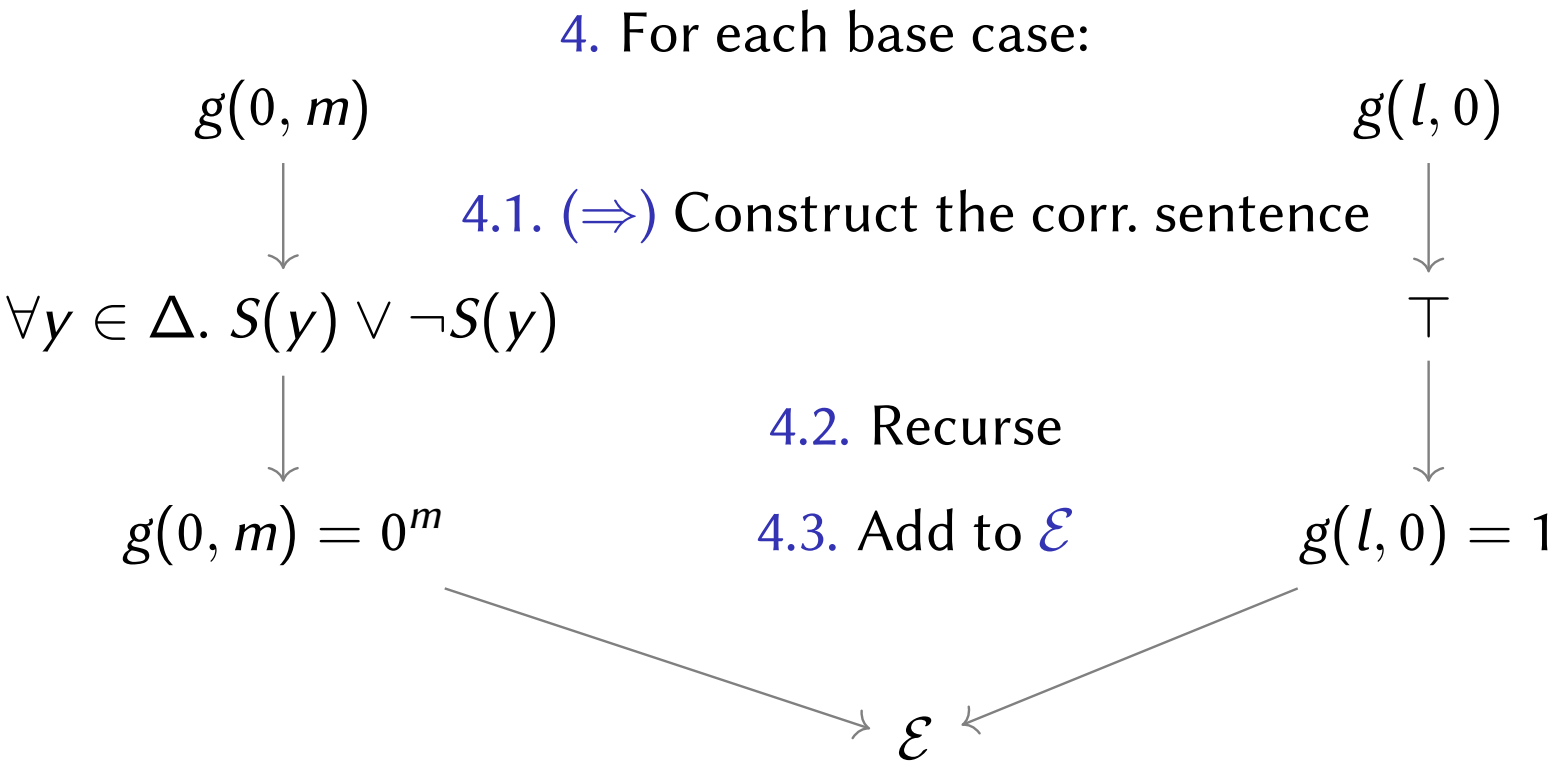

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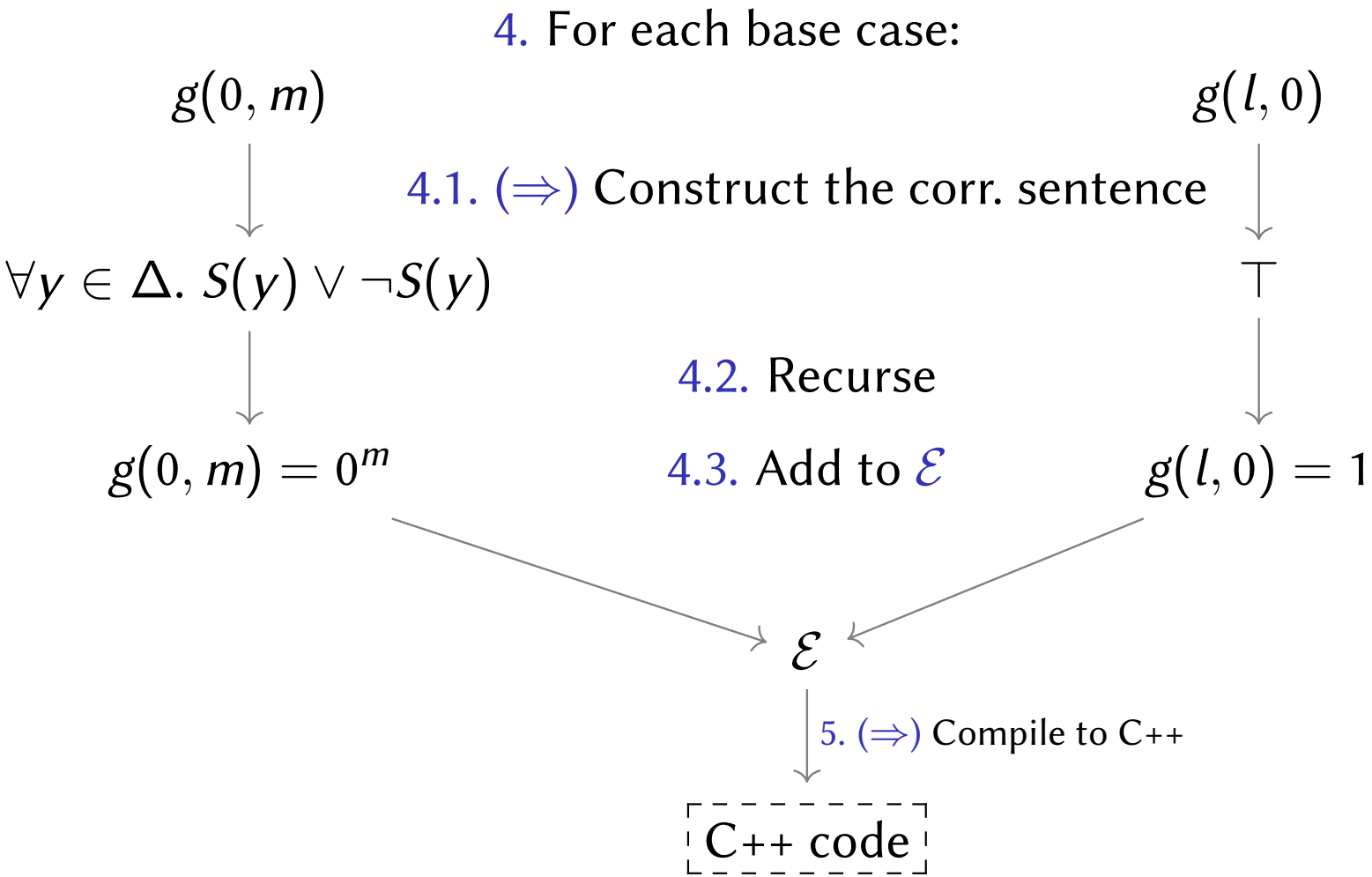

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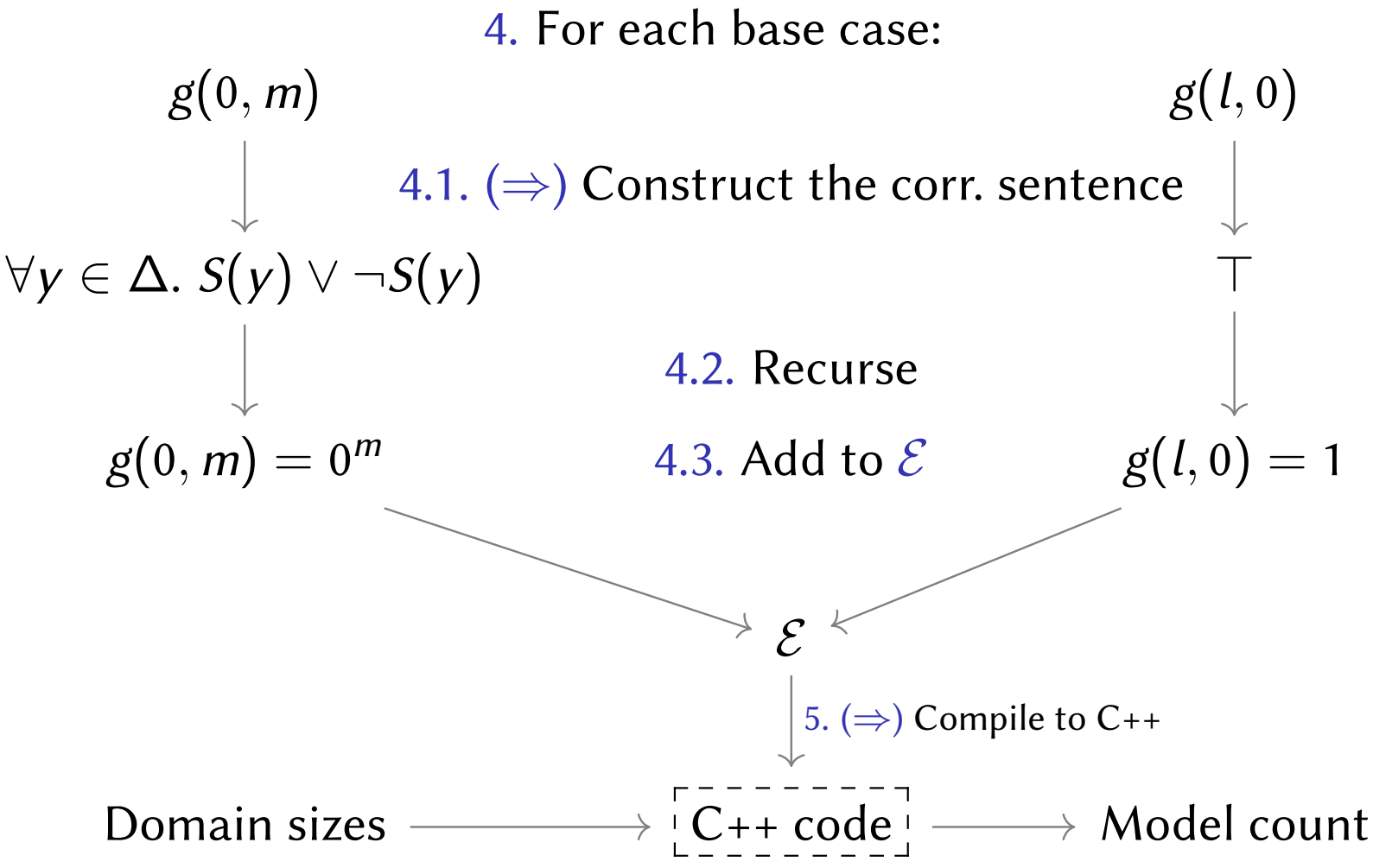


Finding (a Sufficient Set of) Base Cases

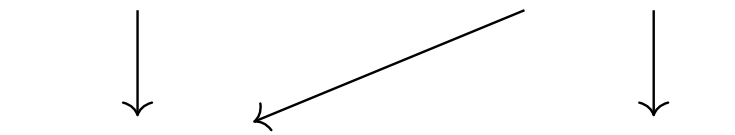
Outline

1. For every function call:
   - 1.1 For every argument of the form $var - const$:
     - 1.1.1 Replace the signature parameter with $0, 1, \ldots, const - 1$
   - 1.2 For every argument of the form $const$:
     - 1.2.1 Replace the corresponding signature parameter with $const$

Example

The signature of g is $g(l, m)$.

Function calls: $g(l-1, m)$ $g(l-1, m-1)$



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No Infinite Cycles

Theorem

The evaluation of a recursive function always terminates.

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Proof (hints).

- There exists a topological ordering of functions
- All function calls follow the structure from the previous slide
- Some common-sense assumptions about the evaluation order and previous work

□

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From Previous Work (Dilkas and Belle 2023)

- CRANE associates each function f with a sentence ϕ such that CRANE(ϕ) produces the definition of f
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- $g(0, \dots)$ means we need to simplify (1) by assuming $|\Gamma| = 0$
- Result: $\forall y \in \Delta.\, S(y) \vee \neg S(y)$ (Smoothing)

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Function $g(l, m)$:
    if $(l, m) \in \text{Cache}_{g}$ **then return** $\text{Cache}_{g}(l, m)$;
    if $l = 0$ **then return** $g_{0,m}(m)$;
    if $m = 0$ **then return** $g_{l,0}(l)$;
    $r \leftarrow g(l-1, m) + mg(l-1, m-1)$;
    $\text{Cache}_{g}(l, m) \leftarrow r$;
    return r;
Function $f(m, n)$: ...
Function `Main`:
    $(m, n) \leftarrow$ `ParseCommandLineArguments();`
    return $f(m, n)$;

Benchmarks

- Friends & Smokers

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$$(\forall x \in \Gamma.\ \exists y \in \Delta.\ P(x, y)) \land$$
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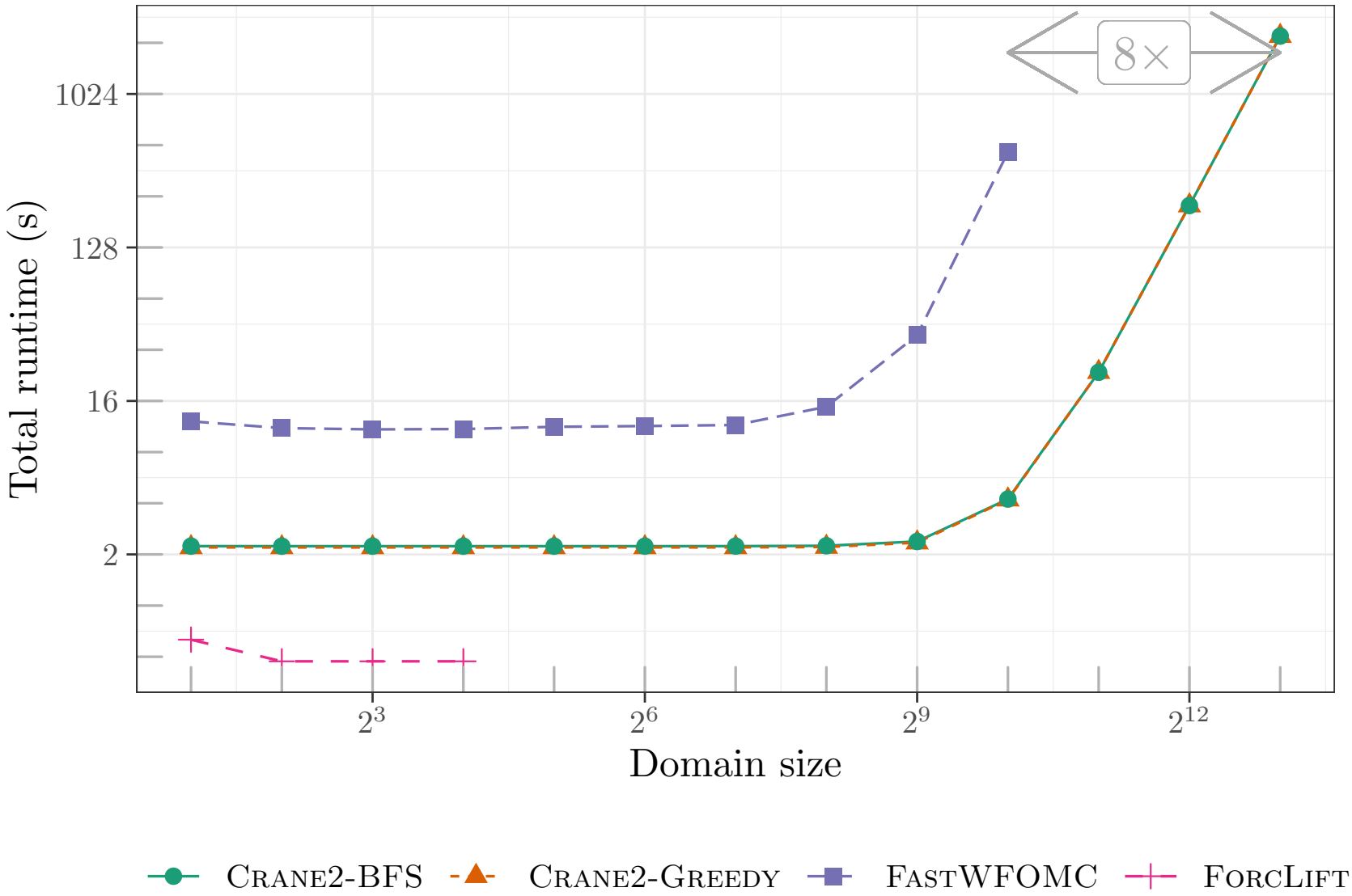

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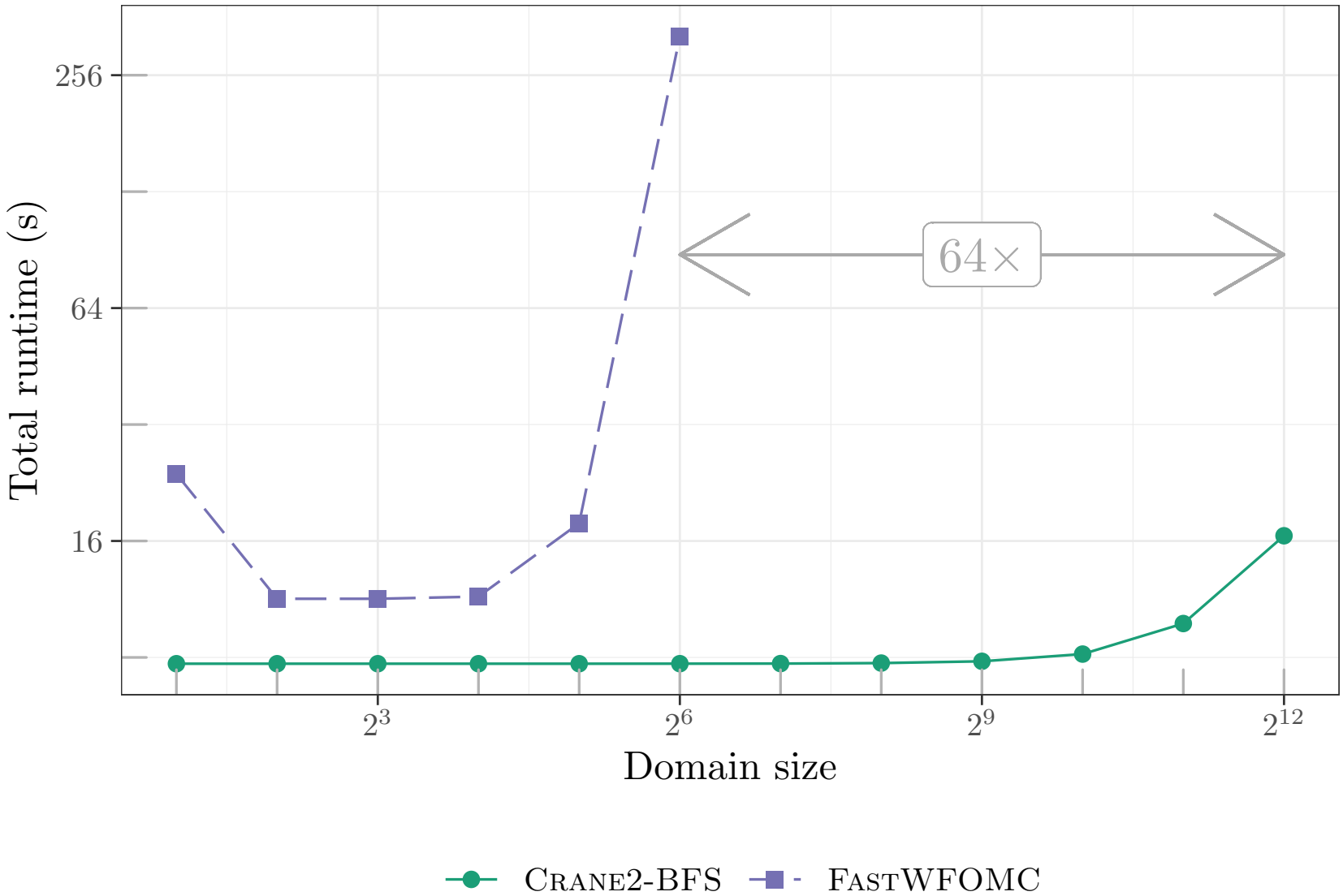
256
64
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Total runtime (s)
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2^3
2^6
2^9
2^{12}
Domain size
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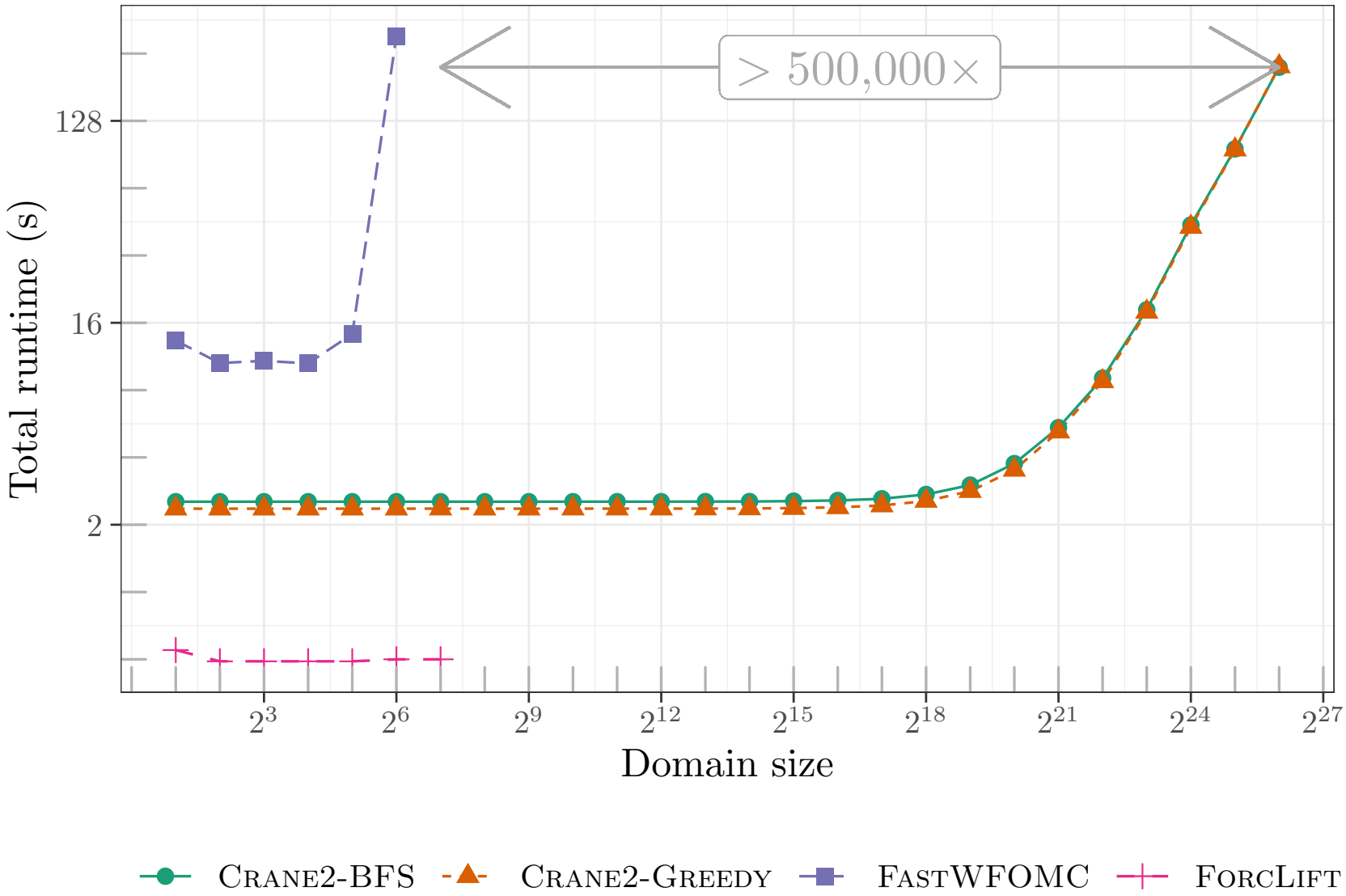
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Summary & Future Work

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Usability: compilation to C++ programs

Scalability compared to other FOMC algorithms

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Future Work

- Support for weighted counting (trivial)
- Experiments on a large set of benchmarks
- Completeness for fragments of first-order logic
- Fine-grained complexity