Weighted Model Counting with Conditional Weights for Bayesian Networks

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Abstract

Weighted model counting (WMC) has emerged as the unifying inference mechanism across many (probabilistic) domains. Encoding an inference problem as an instance of WMC typically necessitates adding extra literals and clauses. This is partly so because the predominant definition of WMC assigns weights to models based on weights on literals, and this severely restricts what probability distributions can be represented. We develop a measure-theoretic perspective on WMC and propose a way to encode conditional weights on literals analogously to conditional probabilities. This representation can be as succinct as standard WMC with weights on literals but can also expand as needed to represent probability distributions with less structure. To demonstrate the performance benefits of conditional weights over the addition of extra literals, we develop a new WMC encoding for Bayesian networks and adapt a state-of-the-art WMC algorithm ADDMC to the new format. Our experiments show that the new encoding significantly improves the performance of the algorithm on most benchmark instances.

1 INTRODUCTION

Weighted model counting (WMC), i.e., an extension of model counting (#SAT) that assigns a weight to every model [Sang et al., 2005], has emerged as one of the most dominant and competitive approaches for handling inference tasks in a wide range of formalisms including Bayesian networks [Sang et al., 2005, Darwiche, 2009], probabilistic graphical models more generally [Choi et al., 2013], and probabilistic programs [Fierens et al., 2015, Holtzen et al., 2020]. Over the last fifteen years, WMC has been extended and generalised in many ways, e.g., to handle continuous probability distributions [Belle et al., 2015], first-order probabilistic theories [Van den Broeck et al., 2011, Gogate and Domingos, 2016], and infinite domains [Belle, 2017]. Furthermore, by generalising the notion of weights to an arbitrary semiring, a range of other problems are also captured [Kimmig et al., 2017]. Exact WMC solvers typically rely on either knowledge compilation [Oztok and Darwiche, 2015, Lagniez and Marquis, 2017] or exhaustive DPLL search [Sang et al., 2005], whereas approximate solvers work by sampling [Chakraborty et al., 2014] and performing local search [Wei and Selman, 2005].

The most well-known version of WMC assigns weights to models based on weights on literals, i.e., the weight of a model is the product of the weights of all literals in it. This simplification is motivated by the fact that the number of models scales exponentially with the number of atoms, so listing the weight of every model is intractable. However, this also severely restricts what probability distributions can be represented. A common way to overcome this limitation is by adding more literals. While we show that this is always possible, we demonstrate that it can be significantly more efficient to encode weights in a more flexible format instead.

After briefly reviewing the background in Section 2, in Section 3 we describe three equivalent perspectives on the subject based on logic, set theory, and Boolean algebras. Furthermore, we describe the space of functions on Boolean algebras and various operations on those functions. Section 4 introduces WMC as the problem of computing the value of a measure on a Boolean algebra. We show that not all measures can be represented using literal-based WMC, but all Boolean algebras can be extended to make any measure representable in such a manner.

This new perspective allows us to not only encode any discrete probability distribution but also improve inference speed. In Section 5 we demonstrate this by developing a new WMC encoding for Bayesian networks that uses *conditional weights* on literals (in the spirit of conditional probabilities) that have literal-based WMC as a special case. We prove

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the correctness of the encoding and show how a state-of-the-art WMC solver ADDMC [Dudek et al., 2020a] can be adapted to the new format. ADDMC is a recently-proposed algorithm for WMC based on manipulating functions on Boolean algebras using an efficient representation for such functions known as algebraic decision diagrams (ADDs) [Bahar et al., 1997]. ADDMC was already shown to be capable of solving instances other solvers fail at and being the fastest solver on the largest number of instances [Dudek et al., 2020a]. Our experiments in Section 6 focus on further improving the performance of ADDMC on instances that originate from Bayesian networks. We show how our new encoding improves inference on the vast majority of benchmark instances, often by one or two orders of magnitude. We explain the performance benefits by showing how our encoding has asymptotically fewer variables and ADDs.

2 RELATED WORK

Performing inference on Bayesian networks by encoding them into instances of WMC is a well-established idea with a history of almost twenty years. Five encodings have been proposed so far (we will identify them based on the initials of authors as well as publications years): `d02` [Darwiche, 2002], `sbk05` [Sang et al., 2005], `cd05` [Chavira and Darwiche, 2005], `cd06` [Chavira and Darwiche, 2006], and `bklm16` [Bart et al., 2016][1]. Below we summarise the observed performance differences among them.

Sang et al. [2005] claim that `sbk05` is a smaller encoding than `d02` with respect to both the number of clauses and the number of variables but provide no experimental comparison. Chavira and Darwiche [2005] compare `cd05` with `d02` by measuring the time it takes to compile either encoding into an arithmetic circuit. They show that `cd05` always compiles faster and results in a smaller arithmetic circuit (as measured by the number of edges). In their subsequent paper, the same authors perform two sets of experiments (that are relevant to this summary) [Chavira and Darwiche, 2006]. First, they compile `cd05` and `cd06` encodings into d-DNNF (i.e., deterministic decomposable negation normal form [Darwiche, 2001]), measuring both compilation time and numbers of edges in the d-DNNF diagram. The results are mostly in favour of `cd06`. Second, they compare the inference time of `sbk05` run with Cachet [Sang et al., 2004] with the compile times of `cd05` and `cd06`, but only on five (types of) instances. In these experiments, `cd06` is always faster than `cd05`, while the comparison with `sbk05` is mixed. The performance difference between `sbk05` and `cd05` is even harder to judge: `sbk05` is better on three out of five instances and worse on the remaining two. Finally, Bart et al. [2016] introduce `bklm16` and show that it has both fewer variables and fewer clauses than `cd06`. Their experiments show `bklm16` to be superior to `cd06` with respect to both compilation time and encoding size when both are compiled using c2d[2] [Darwiche, 2004] but inferior to `cd06` when `cd06` is compiled using Ace[3] (which still uses c2d but considers the structure of the Bayesian network along with its encoding). Our experiments in Section 6 confirm some of the findings outlined in this section while also showing that the performance of each encoding depends on the WMC algorithm in use, and smaller encodings are not necessarily faster.

While in this paper we focus on measure-theoretic foundations and propose a new encoding for Bayesian networks, a related line of work [Dilkas and Belle, 2021]—motivated by the same issue of WMC encodings having both more variables and more clauses as a consequence of having to conform to an unnecessarily restrictive format—shows how these variables and clauses can be removed from (most) already-existing Bayesian network encodings.

3 BOOLEAN ALGEBRAS, POWER SETS, AND PROPOSITIONAL LOGIC

In this section, we give a brief introduction to two alternative ways to think about logical constructs such as models and formulas. Let us consider a simple example of a propositional logic $\mathcal{L}$ with only two atoms a and b, and let $U = \{a, b\}$. Then 2^U, the power set of U, is the set of all models of $\mathcal{L}$, and 2^{2^U} is the set of all formulas. These sets can also be represented as Boolean algebras (e.g., using the syntax $(2^{2^U}, \wedge, \vee, \neg, \bot, \top)$) with a partial order $\leq$ that corresponds to set inclusion $\subseteq$—see Table 1 for examples of how various elements can be represented in both notations. Most importantly, note that the word *atom* has completely different meanings in logic and Boolean algebras. An atom in $\mathcal{L}$ is an atomic formula, i.e., an element of U, whereas an atom in a Boolean algebra is (in set-theoretic terms) a singleton set. For instance, an atom in 2^{2^U} corresponds to a model of $\mathcal{L}$, i.e., an element of 2^U. Unless referring specifically to a logic, we will use the algebraic definition of an atom and refer to logical atoms as *variables*. In the rest of the paper, for any set U, we will use set-theoretic notation for 2^U and Boolean-algebraic notation for 2^{2^U}, except for (Boolean) atoms in 2^{2^U} that are denoted as $\{x\}$ for some model $x \in 2^U$.

3.1 FUNCTIONS ON BOOLEAN ALGEBRAS

We also consider the space of all functions from any Boolean algebra to $\mathbb{R}_{\geq 0}$ together with some operations on those functions. They will be instrumental in defining WMC as a

[1] Vomlel and Tichavský [2013] also propose an encoding, but only for networks of a particular bipartite structure and without any evaluation.

[2] `http://reasoning.cs.ucla.edu/c2d/`

[3] `http://reasoning.cs.ucla.edu/ace/`

Table 1: Notation for a logic with two atoms. The elements in both columns are listed in the same order.

Name in logic	Boolean-algebraic notation	Set-theoretic notation
Atoms (elements of U)	a, b	a, b
Models (elements of 2^U)	$\neg a \wedge \neg b, a \wedge \neg b, \neg a \wedge b, a \wedge b$	$\emptyset, \{a\}, \{b\}, \{a, b\}$
Formulas (elements of 2^{2^U})	$\top$	$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
	$\neg a \vee \neg b, a \to b$	$\{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{b\}, \{a, b\}\}$
	$b \to a, a \vee b$	$\{\emptyset, \{a\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}$
	$\neg b, \neg a, a \leftrightarrow b$	$\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a, b\}\}$
	$(a \wedge \neg b) \vee (b \wedge \neg a), a, b$	$\{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}$
	$\neg a \wedge \neg b, a \wedge \neg b, \neg a \wedge b, a \wedge b$	$\{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}$
	$\bot$	$\emptyset$

measure in Section 4 and can be efficiently represented using ADDs. Furthermore, all of the operations are supported by CUDD [Somenzi, 2015]—a package used by ADDMC for ADD manipulation [Dudek et al., 2020a]. The definitions of multiplication and projection are as defined by Dudek et al. [2020a], while others are new.

Definition 1 (Operations on functions)**.** Let $\alpha\colon 2^X \to \mathbb{R}_{\geq 0}$ and $\beta\colon 2^Y \to \mathbb{R}_{\geq 0}$ be functions, $p \in \mathbb{R}_{\geq 0}$, and $x \in X$. We define the following operations:

Addition: $\alpha+\beta\colon 2^{X\cup Y} \to \mathbb{R}_{\geq 0}$ is such that $(\alpha+\beta)(T) = \alpha(T \cap X) + \beta(T \cap Y)$ for all $T \in 2^{X\cup Y}$.

Multiplication: $\alpha \cdot \beta\colon 2^{X\cup Y} \to \mathbb{R}_{\geq 0}$ is such that $(\alpha \cdot \beta)(T) = \alpha(T \cap X) \cdot \beta(T \cap Y)$ for all $T \in 2^{X\cup Y}$.

Scalar multiplication: $p\alpha\colon 2^X \to \mathbb{R}_{\geq 0}$ is such that $(p\alpha)(T) = p \cdot \alpha(T)$ for all $T \in 2^X$.

Complement: $\overline{\alpha}\colon 2^X \to \mathbb{R}_{\geq 0}$ is such that $\overline{\alpha}(T) = 1 - \alpha(T)$ for all $T \in 2^X$.

Projection: $\exists_x \alpha\colon 2^{X\setminus\{x\}} \to \mathbb{R}_{\geq 0}$ is such that $(\exists_x \alpha)(T) = \alpha(T) + \alpha(T \cup \{x\})$ for all $T \in 2^{X\setminus\{x\}}$. For any $Z = \{z_1, \ldots, z_n\} \subseteq X$, we write $\exists_Z$ to mean $\exists_{z_1} \ldots \exists_{z_n}$.

In summary, addition, multiplication, and scalar multiplication are defined pointwise, while complement and projection interact with the algebraic structure of the domains 2^X and 2^Y. Specifically, note that both addition and multiplication are both associative and commutative. We end the discussion on function spaces by defining several special functions: unit $1\colon 2^\emptyset \to \mathbb{R}_{\geq 0}$ defined as $1(\emptyset) = 1$, zero $0\colon 2^\emptyset \to \mathbb{R}_{\geq 0}$ defined as $0(\emptyset) = 0$, and function $[a]\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$ defined as $[a](\emptyset) = 0$, $[a](\{a\}) = 1$ for any a. Henceforth, for any function $\alpha\colon 2^X \to \mathbb{R}_{\geq 0}$ and any set T, we will write $\alpha(T)$ to mean $\alpha(T \cap X)$.

4 WMC AS A MEASURE ON A BOOLEAN ALGEBRA

In this section, we introduce an alternative definition of WMC and demonstrate how it relates to the standard one. Let U be a set. A *measure* is a function $\mu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ such that $\mu(\bot) = 0$, and $\mu(a \vee b) = \mu(a) + \mu(b)$ for all $a, b \in 2^{2^U}$ whenever $a \wedge b = \bot$ [Gaifman, 1964, Jech, 1997]. A *weight function* is a function $\nu\colon 2^U \to \mathbb{R}_{\geq 0}$. A weight function is *factored* if $\nu = \prod_{x\in U} \nu_x$ for some functions $\nu_x\colon 2^{\{x\}} \to \mathbb{R}_{\geq 0}$, $x \in U$. We say that a weight function $\nu\colon 2^U \to \mathbb{R}_{\geq 0}$ *induces* a measure $\mu_\nu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ if $\mu_\nu(x) = \sum_{\{u\}\leq x} \nu(u)$.

Theorem 1. *The function μ_ν is a measure.*

Finally, a measure $\mu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ is *factorable* if there exists a factored weight function $\nu\colon 2^U \to \mathbb{R}_{\geq 0}$ that induces μ. In this formulation, WMC corresponds to the process of calculating the value of $\mu_\nu(x)$ for some $x \in 2^{2^U}$ with a given definition of ν.

Relation to the classical (logic-based) view of WMC. Let $\mathcal{L}$ be a propositional logic with two atoms a and b as in Section 3 and $w\colon \{a, b, \neg a, \neg b\} \to \mathbb{R}_{\geq 0}$ a weight function defined as $w(a) = 0.3$, $w(\neg a) = 0.7$, $w(b) = 0.2$, $w(\neg b) = 0.8$. Furthermore, let Δ be a theory in $\mathcal{L}$ with a sole axiom a. Then Δ has two models: $\{a, b\}$ and $\{a, \neg b\}$ and its WMC [Chavira and Darwiche, 2008] is

$$\begin{aligned} \mathrm{WMC}(\Delta) &= \sum_{\omega\models\Delta} \prod_{\omega\models l} w(l) \\ &= w(a)w(b) + w(a)w(\neg b) = 0.3. \end{aligned} \tag{1}$$

Alternatively, we can define $\nu_a\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$ as $\nu_a(\{a\}) = 0.3$, $\nu_a(\emptyset) = 0.7$ and $\nu_b\colon 2^{\{b\}} \to \mathbb{R}_{\geq 0}$ as $\nu_b(\{b\}) = 0.2$, $\nu_b(\emptyset) = 0.8$. Let μ be the measure on 2^{2^U} induced by $\nu = \nu_a \cdot \nu_b$. Then, equivalently to Eq. (1), we can write

$$\begin{aligned} \mu(a) &= \nu(\{a, b\}) + \nu(\{a\}) \\ &= \nu_a(\{a\})\nu_b(\{b\}) + \nu_a(\{a\})\nu_b(\emptyset) = 0.3. \end{aligned}$$

Thus, one can equivalently think of WMC as summing over models of a theory or over atoms below an element of a Boolean algebra.

4.1 NOT ALL MEASURES ARE FACTORABLE

Using this new definition of WMC, we can show that WMC with weights defined on literals is only able to capture a subset of all possible measures on a Boolean algebra. This can be demonstrated with a simple example.

Example 1. Let $U = \{a, b\}$ be a set of atoms and $\mu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$ a measure defined as $\mu(a \wedge b) = 0.72$, $\mu(a \wedge \neg b) = 0.18$, $\mu(\neg a \wedge b) = 0.07$, $\mu(\neg a \wedge \neg b) = 0.03$.[4] If μ could be represented using literal-weight (factored) WMC, we would have to find two weight functions $\nu_a\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$ and $\nu_b\colon 2^{\{b\}} \to \mathbb{R}_{\geq 0}$ such that $\nu = \nu_a \cdot \nu_b$ induces μ, i.e., ν_a and ν_b would have to satisfy this system of equations:

$$\begin{aligned} \nu_a(\{a\}) \cdot \nu_b(\{b\}) &= 0.72 \\ \nu_a(\{a\}) \cdot \nu_b(\emptyset) &= 0.18 \\ \nu_a(\emptyset) \cdot \nu_b(\{b\}) &= 0.07 \\ \nu_a(\emptyset) \cdot \nu_b(\emptyset) &= 0.03, \end{aligned}$$

which has no solutions.

Alternatively, we can let b depend on a and consider weight functions $\nu_a\colon 2^{\{a\}} \to \mathbb{R}_{\geq 0}$ and $\nu_b\colon 2^{\{a,b\}} \to \mathbb{R}_{\geq 0}$ defined as $\nu_a(\{a\}) = 0.9$, $\nu_a(\emptyset) = 0.1$, and $\nu_b(\{a, b\}) = 0.8$, $\nu_b(\{a\}) = 0.2$, $\nu_b(\{b\}) = 0.7$, $\nu_b(\emptyset) = 0.3$. One can easily check that with these definitions ν indeed induces μ.

Note that in this case, we chose to interpret ν_b as $\Pr(b \mid a)$ while—with a different definition of ν_b that represents the joint probability distribution $\Pr(a, b)$—ν_b by itself could induce μ. In general, however, factorising the full weight function into several smaller functions often results in weight functions with smaller domains which leads to increased efficiency and decreased memory usage [Dudek et al., 2020a]. We can easily generalise this example further.

Theorem 2. *For any set U such that $|U| \geq 2$, there exists a non-factorable measure $2^{2^U} \to \mathbb{R}_{\geq 0}$.*

Since many measures of interest may not be factorable, a well-known way to encode them into instances of WMC is by adding more literals [Chavira and Darwiche, 2008]. We can use the measure-theoretic perspective on WMC to show that this is always possible, however, as ensuing sections will demonstrate, it can make the inference task much harder in practice.[5]

[4] The value of μ on any other element of 2^{2^U} can be deduced from the definition of a measure.

[5] The proofs of this and other theoretical results can be found in the supplementary material.

Theorem 3. *For any set U and measure $\mu\colon 2^{2^U} \to \mathbb{R}_{\geq 0}$, there exists a set $V \supseteq U$, a factorable measure $\mu'\colon 2^{2^V} \to \mathbb{R}_{\geq 0}$, and a formula $f \in 2^{2^V}$ such that $\mu(x) = \mu'(x \wedge f)$ for all formulas $x \in 2^{2^U}$.*

5 ENCODING BAYESIAN NETWORKS USING CONDITIONAL WEIGHTS

In this section, we describe a way to encode Bayesian networks into WMC without restricting oneself to factorable measures and thus having to add extra variables. We will refer to it as `cw`. A Bayesian network is a directed acyclic graph with random variables as vertices that defines a probability distribution over them. Let $\mathcal{V}$ denote this set of random variables. For any random variable $X \in \mathcal{V}$, let $\operatorname{im} X$ denote its set of values and $\operatorname{pa}(X)$ its set of parents. The full probability distribution is then equal to $\prod_{X \in \mathcal{V}} \Pr(X \mid \operatorname{pa}(X))$. For discrete Bayesian networks (and we only consider discrete networks in this paper), each factor of this product can be represented by a CPT. See Fig. 1 for an example Bayesian network that we will refer to throughout this section. For this network, $\mathcal{V} = \{W, F, T\}$, $\operatorname{pa}(W) = \emptyset$, $\operatorname{pa}(F) = \operatorname{pa}(T) = \{W\}$, $\operatorname{im} W = \operatorname{im} F = \{0, 1\}$, and $\operatorname{im} T = \{l, m, h\}$.

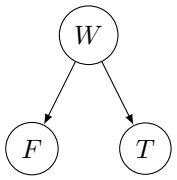
Definition 2 (Indicator variables)**.** Let $X \in \mathcal{V}$ be a random variable. If X is binary (i.e., $|\operatorname{im} X| = 2$), we can arbitrary identify one of the values as 1 and the other one as 0 (i.e, $\operatorname{im} X \cong \{0, 1\}$). Then X can be represented by a single *indicator variable* $\lambda_{X=1}$. For notational simplicity, for any set S, we write $\lambda_{X=0} \in S$ or $S = \{\lambda_{X=0}, \dots\}$ to mean $\lambda_{X=1} \notin S$.

On the other hand, if X is not binary, we represent X with $|\operatorname{im} X|$ indicator variables, one for each value. We let

$$\mathcal{E}(X) = \begin{cases} \{\lambda_{X=1}\} & \text{if } |\operatorname{im} X| = 2 \\ \{\lambda_{X=x} \mid x \in \operatorname{im} X\} & \text{otherwise.} \end{cases}$$

denote the set of indicator variables for X and $\mathcal{E}^*(X) = \mathcal{E}(X) \cup \bigcup_{Y \in \operatorname{pa}(X)} \mathcal{E}(Y)$ denote the set of indicator variables for X and its parents in the Bayesian network. Finally, let $U = \bigcup_{X \in \mathcal{V}} \mathcal{E}(X)$ denote the set of all indicator variables for all random variables in the Bayesian network. For example, in the Bayesian network from Fig. 1, $\mathcal{E}^*(T) = \{\lambda_{T=l}, \lambda_{T=m}, \lambda_{T=h}, \lambda_{W=1}\}$.

Algorithm 1 shows how a Bayesian network with vertices $\mathcal{V}$ can be represented as a weight function $\phi\colon 2^U \to \mathbb{R}_{\geq 0}$. The algorithm begins with the unit function and multiplies it by $\mathrm{CPT}_X\colon 2^{\mathcal{E}^*(X)} \to \mathbb{R}_{\geq 0}$ for each random variable $X \in \mathcal{V}$. We call each such function a *conditional weight function* as it represents a conditional probability distribution. However, the distinction is primarily a semantic one: a function $2^{\{a,b\}} \to \mathbb{R}_{\geq 0}$ can represent $\Pr(a \mid b)$, $\Pr(b \mid a)$, or something else entirely, e.g., $\Pr(a \wedge b)$, $\Pr(a \vee b)$, etc.



w	$\Pr(W = w)$
1	0.5
0	0.5

w	f	$\Pr(F = f \mid W = w)$
1	1	0.6
1	0	0.4
0	1	0.1
0	0	0.9

w	t	$\Pr(T = t \mid W = w)$
1	l	0.2
1	m	0.4
1	h	0.4
0	l	0.6
0	m	0.3
0	h	0.1

Figure 1: An example Bayesian network with its CPTs.

Algorithm 1: Encoding a Bayesian network.

Data: vertices $\mathcal{V}$, probability distribution Pr
Result: $\phi\colon 2^U \to \mathbb{R}_{\geq 0}$
$\phi \leftarrow 1$;
for $X \in \mathcal{V}$ **do**
  let $\mathrm{pa}(X) = \{Y_1, \ldots, Y_n\}$;
  $\mathrm{CPT}_X \leftarrow 0$;
  if $|\operatorname{im} X| = 2$ **then**
    for $(y_i)_{i=1}^n \in \prod_{i=1}^n \operatorname{im} Y_i$ **do**
      $p_1 \leftarrow \Pr(X = 1 \mid y_1, \ldots, y_n)$;
      $p_0 \leftarrow \Pr(X \neq 1 \mid y_1, \ldots, y_n)$;
      $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X$
        $+p_1[\lambda_{X=1}] \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$
        $+p_0\overline{[\lambda_{X=1}]} \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$;
  else
    let $\operatorname{im} X = \{x_1, \ldots, x_m\}$;
    for $x \in \operatorname{im} X$ **and** $(y_i)_{i=1}^n \in \prod_{i=1}^n \operatorname{im} Y_i$ **do**
      $p_x \leftarrow \Pr(X = x \mid y_1, \ldots, y_n)$;
      $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X$
        $+p_x[\lambda_{X=x}] \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$
        $+\overline{[\lambda_{X=x}]} \cdot \prod_{i=1}^n [\lambda_{Y_i=y_i}]$;
    $\mathrm{CPT}_X \leftarrow \mathrm{CPT}_X \cdot (\sum_{i=1}^m [\lambda_{X=x_i}])$
      $\cdot \prod_{i=1}^m \prod_{j=i+1}^m (\overline{[\lambda_{X=x_i}]} + \overline{[\lambda_{X=x_j}]})$;
  $\phi \leftarrow \phi \cdot \mathrm{CPT}_X$;
return ϕ;

For a binary random variable X, CPT_X is simply a sum of smaller functions, one for each row of the CPT. If X has more than two values, we also multiply CPT_X by 'clause' functions that restrict the value of $\phi(T)$ to zero whenever $|\mathcal{E}(X) \cap T| \neq 1$, i.e., we add mutual exclusivity constraints that ensure that each random variable is associated with exactly one value. Note that Chavira and Darwiche [2007] use the same ADD representation of CPTs for their compilation algorithm based on variable elimination. For the example Bayesian network in Fig. 1, we get:

$$\begin{aligned}\mathrm{CPT}_\mathrm{F} &= 0.6[\lambda_{F=1}] \cdot [\lambda_{W=1}] + 0.4[\lambda_{F=0}] \cdot [\lambda_{W=1}] \\ &\quad + 0.1[\lambda_{F=1}] \cdot [\lambda_{W=0}] + 0.9[\lambda_{F=0}] \cdot [\lambda_{W=0}], \\ \mathrm{CPT}_\mathrm{T} &= ([\lambda_{T=l}] + [\lambda_{T=m}] + [\lambda_{T=h}]) \\ &\quad \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=m}]}) \cdot (\overline{[\lambda_{T=l}]} + \overline{[\lambda_{T=h}]}) \\ &\quad \cdot (\overline{[\lambda_{T=m}]} + \overline{[\lambda_{T=h}]}) \cdot \ldots.\end{aligned}$$

5.1 CORRECTNESS

Algorithm 1 produces a function with a Boolean algebra as its domain. This function can be represented by an ADD [Bahar et al., 1997]. ADDMC takes an ADD $\psi\colon 2^U \to \mathbb{R}_{\geq 0}$ (expressed as a product of smaller ADDs) and returns $(\exists_U \psi)(\emptyset)$ [Dudek et al., 2020a]. In this section, we prove that the function ϕ produced by Algorithm 1 can be used by ADDMC to correctly compute any marginal probability of the Bayesian network that was encoded as ϕ.[6] We begin with Lemma 1 which shows that any conditional weight function produces the right answer when given a valid encoding of variable-value assignments relevant to the CPT.

Lemma 1. *Let $X \in \mathcal{V}$ be a random variable with parents $\mathrm{pa}(X) = \{Y_1, \ldots, Y_n\}$. Then $\mathrm{CPT}_X\colon 2^{\mathcal{E}^*(X)} \to \mathbb{R}_{\geq 0}$ is such that for any $x \in \operatorname{im} X$ and $(y_1, \ldots, y_n) \in \prod_{i=1}^n \operatorname{im} Y_i$,*

$$\mathrm{CPT}_X(T) = \Pr(X = x \mid Y_1 = y_1, \ldots, Y_n = y_n),$$

where $T = \{\lambda_{X=x}\} \cup \{\lambda_{Y_i=y_i} \mid i = 1, \ldots, n\}$.

Now, Lemma 2 shows that ϕ represents the full probability distribution of the Bayesian network, i.e., it gives the right probabilities for the right inputs and zero otherwise.

Lemma 2. *Let $\mathcal{V} = \{X_1, \ldots, X_n\}$. Then*

$$\phi(T) = \begin{cases} \Pr(x_1, \ldots, x_n) & \text{if } T = \{\lambda_{X_i=x_i}\}_{i=1}^n \text{ for} \\ & \text{some } (x_i)_{i=1}^n \in \prod_{i=1}^n \operatorname{im} X_i \\ 0 & \text{otherwise,} \end{cases}$$

for all $T \in 2^U$.

[6] Note that it can just as well compute *any* probability expressed using the random variables in $\mathcal{V}$.

We end with Theorem 4 that shows how ϕ can be combined with an encoding of a single variable-value assignment so that ADDMC [Dudek et al., 2020a] would compute its marginal probability.

Theorem 4. *For any $X \in \mathcal{V}$ and $x \in \operatorname{im} X$,*

$$(\exists_U(\phi \cdot [\lambda_{X=x}]))(\emptyset) = \Pr(X = x).$$

5.2 TEXTUAL REPRESENTATION

Algorithm 1 encodes a Bayesian network into a function on a Boolean algebra, but how does it relate to the standard interpretation of a WMC encoding as a formula in conjunctive normal form (CNF) together with a collection of weights? The factors of ϕ that restrict the values of indicator variables for non-binary random variables are already expressed as a product of sums of 0/1-valued functions, i.e., a kind of CNF. Disregarding these functions, each conditional weight function CPT_X is represented by a sum with a term for every subset of $\mathcal{E}^*(X)$. To encode these terms, we introduce *extended weight clauses* to the WMC format used by Cachet [Sang et al., 2004]. For instance, here is a representation of the Bayesian network from Fig. 1:

$$\begin{array}{llllll}
\lambda_{T=l} & \lambda_{T=m} & \lambda_{T=h} & & 0 \\
 & -\lambda_{T=l} & -\lambda_{T=m} & & 0 \\
 & -\lambda_{T=l} & -\lambda_{T=h} & & 0 \\
 & -\lambda_{T=m} & -\lambda_{T=h} & & 0 \\
w & \lambda_{W=1} & & 0.5 & 0.5 \\
w & \lambda_{F=1} & \lambda_{W=1} & 0.6 & 0.4 \\
w & \lambda_{F=1} & -\lambda_{W=1} & 0.1 & 0.9 \\
w & \lambda_{T=l} & \lambda_{W=1} & 0.2 & 1 \\
w & \lambda_{T=m} & \lambda_{W=1} & 0.4 & 1 \\
w & \lambda_{T=h} & \lambda_{W=1} & 0.4 & 1 \\
w & \lambda_{T=l} & -\lambda_{W=1} & 0.6 & 1 \\
w & \lambda_{T=m} & -\lambda_{W=1} & 0.3 & 1 \\
w & \lambda_{T=h} & -\lambda_{W=1} & 0.1 & 1
\end{array}$$

where each indicator variable is eventually replaced with a unique positive integer. Each line prefixed with a w can be split into four parts: the 'main' variable (always not negated), conditions (possibly none), and two weights. For example, the line

$$w \quad \lambda_{T=m} \quad -\lambda_{W=1} \quad 0.3 \quad 1$$

encodes the function $0.3[\lambda_{T=m}] \cdot \overline{[\lambda_{W=1}]} + 1\overline{[\lambda_{T=m}]} \cdot \overline{[\lambda_{W=1}]}$ and can be interpreted as defining two conditional weights: $\nu(T = m \mid W = 0) = 0.3$, and $\nu(T \neq m \mid W = 0) = 1$, the former of which corresponds to a row in the CPT of T while the latter is artificially added as part of the encoding. In our encoding of Bayesian networks, it is always the case that, in each weight clause, either both weights sum to one, or the second weight is equal to one. Finally, note that the measure induced by these weights is not probabilistic (i.e., $\mu(\top) \neq 1$) by itself, but it becomes probabilistic when combined with the additional clauses that restrict what combinations of indicator variables can co-occur.

5.3 CHANGES TO ADDMC

Here we describe two changes to ADDMC[7] [Dudek et al., 2020a] needed to adapt it to the new format.

First, ADDMC constructs the *primal* (a.k.a. Gaifman) *graph* of the input CNF formula as an aid for the algorithm's heuristics. This graph has as vertices the variables of the formula, and there is an edge between two variables u and v if there is a clause in the formula that contains both u and v. We extend this definition to functions on Boolean algebras, i.e., the factors of ϕ. For any pair of distinct variables $u, v \in U$, we draw an edge between them in the primal graph if there is a function $\alpha\colon 2^X \to \mathbb{R}_{\geq 0}$ that is a factor of ϕ such that $u, v \in X$. For instance, a factor such as CPT_X will enable edges between all distinct pairs of variables in $\mathcal{E}^*(X)$. Second, even though the function ϕ produced by Algorithm 1 is constructed to have 2^U as its domain, sometimes the domain is effectively reduced to 2^V for some $V \subset U$ by the ADD manipulation algorithms that optimise the ADD representation of a function. For a simple example, consider $\alpha : 2^{\{a\}} \to \mathbb{R}_{\geq 0}$ defined as $\alpha(\{a\}) = \alpha(\emptyset) = 0.5$. Then α can be reduced to $\alpha'\colon 2^\emptyset \to \mathbb{R}_{\geq 0}$ defined as $\alpha'(\emptyset) = 0.5$. To compensate for these reductions, for the original WMC format with a weight function $w\colon U \cup \{\neg u \mid u \in U\} \to \mathbb{R}_{\geq 0}$, ADDMC would multiply its computed answer by $\prod_{u \in U \setminus V} w(u) + w(\neg u)$. With the new WMC format, we instead multiply the answer by $2^{|U \setminus V|}$. Each 'excluded' variable $u \in U \setminus V$ satisfies two properties: all weights associated with u are equal to 0.5 (otherwise the corresponding CPT would depend on u, and u would not be excluded), and all other CPTs are independent of u (or they may have a trivial dependence, where the probability stays the same if u is replaced with its complement). Thus, the CPT that corresponds to u still multiplies the weight of every atom in the Boolean algebra by 0.5, but the number of atoms under consideration is halved. To correct for this, we multiply the final answer by two for every $u \in U \setminus V$.

6 EXPERIMENTAL RESULTS

We compare the six WMC encodings for Bayesian networks when run with both ADDMC [Dudek et al., 2020a] and the WMC algorithms used in the original papers.[8] We compare the encodings with respect to the total time it takes to encode a Bayesian network, compile it or run a WMC algorithm on it, and extract the (numerical) answer. Note that while all five papers that introduce other encodings include experimental comparisons of encoding size, that is not feasible with ADDMC as even instances that are fully solved in less

[7] https://github.com/vardigroup/ADDMC

[8] Both `cd05` and `cd06` cannot be run with most WMC algorithms including ADDMC because these encodings allow for additional models that the WMC algorithm is supposed to ignore [Chavira and Darwiche, 2005, 2006].

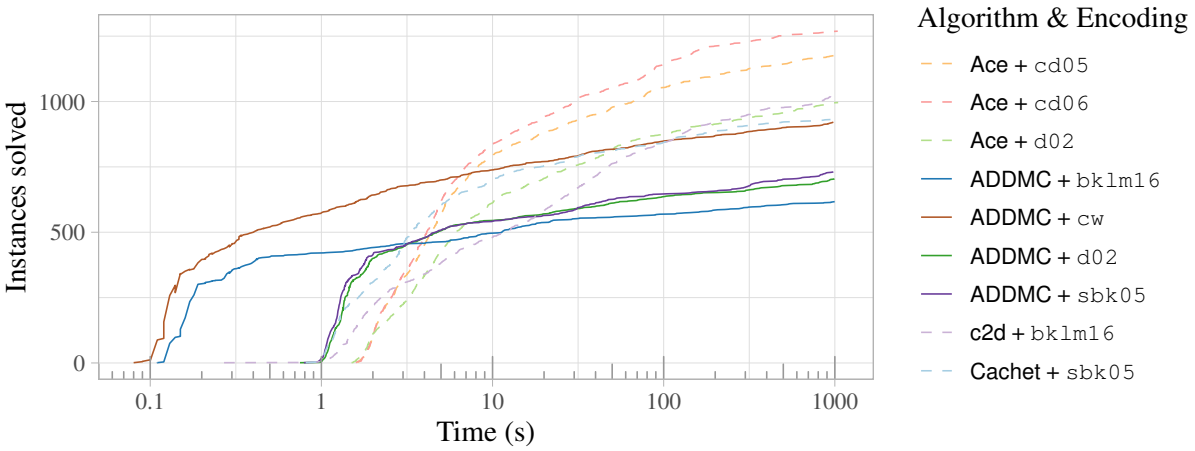


Figure 2: Cumulative numbers of instances solved by combinations of algorithms and encodings over time.

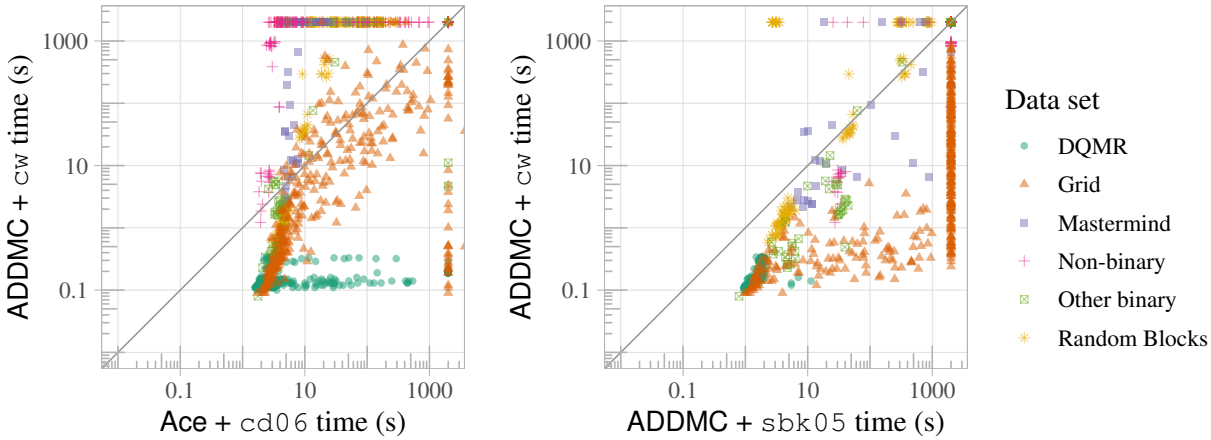


Figure 3: An instance-by-instance comparison between ADDMC + cw and the best overall combination of algorithm and encoding (Ace + cd06, on the left) as well as the second-best encoding for ADDMC (sbk05, on the right).

Table 2: The numbers of instances (out of 1466) solved by each combination of algorithm and encoding (uniquely, faster than others, and in total).

Algorithm & Encoding	Unique	Fastest	Total
Ace + cd05	0	55	1169
Ace + cd06	**34**	218	**1259**
Ace + d02	0	46	993
ADDMC + bklm16	0	29	617
ADDMC + cw	14	**770**	919
ADDMC + d02	0	0	703
ADDMC + sbk05	0	0	729
c2d + bklm16	0	3	1017
Cachet + sbk05	13	229	928

than 0.1 s are too big to build the full ADD within reasonable time and memory limits. The experiments were run on a computing cluster with Intel Xeon Gold 6138 and Intel Xeon E5-2630 processors[9] running Scientific Linux 7 with a 32 GiB memory limit and a 1000 s timeout on both encoding and inference. For inference, we use Ace for cd05 [Chavira and Darwiche, 2005], cd06 [Chavira and Darwiche, 2006], and d02 [Darwiche, 2002]; Cachet[10] [Sang et al., 2004] for sbk05 [Sang et al., 2005]; and c2d [Darwiche, 2004] for compilation and query-dnnf [11] for answer computation for bklm16 [Bart et al., 2016]. For encoding, we use bn2cnf [12] for bklm16, and Ace for all other encodings (except for cw, which is implemented in Python).

[9] Each instance is run on the same processor for all encodings.

[10] https://cs.rochester.edu/u/kautz/Cachet/

[11] http://www.cril.univ-artois.fr/kc/d-DNNF-reasoner.html

[12] http://www.cril.univ-artois.fr/KC/bn2cnf.html

Ace was not used to encode evidence, as preliminary experiments revealed that the evidence-encoding implementation contains bugs that can lead to incorrect answers or a Java exception being thrown on some instances of the data set (and the source code is not publicly available). Instead, we simply list all the evidence as additional clauses in the encoding. Furthermore, to ensure that `bklm16` [Bart et al., 2016] (whether run with ADDMC [Dudek et al., 2020a] or c2d [Darwiche, 2004]) returns correct answers on most instances, we had to disable one of the improvements that `bklm16` brings over `cd06` [Chavira and Darwiche, 2006], namely, the construction of a scaling factor that 'absorbs' one probability from each CDT [Bart et al., 2016]. For realistic benchmark instances, this scaling factor can easily be below 10^{-30}, and thus would require arbitrary-precision floating-point arithmetic to be usable. Even a toy Bayesian network with seven binary independent variables with probabilities 0.1 and 0.9 is enough for bn2cnf to output precisely zero as the scaling factor. We note that this issue likely remained unnoticed because Bart et al. [2016] did not attempt to compute numerical answers in their experiments.

For each Bayesian network, we need to choose a probability to compute. Whenever a Bayesian network comes with an evidence file, we compute the probability of evidence. Otherwise, let X denote the last-mentioned vertex in the Bayesian network. If $\mathsf{true} \in \operatorname{im} X$, then we compute the marginal probability of $X = \mathsf{true}$. Otherwise, we pick the value of X which is listed first and calculate its marginal probability.

For experimental data, we use the Bayesian networks available with Ace and Cachet [Sang et al., 2004], most of which happen to be binary. We classify them into the following seven categories: • DQMR and • Grid networks as described by Sang et al. [2005], • Mastermind, and • Random Blocks from the work of Chavira et al. [2006], • remaining binary Bayesian networks that include Plan Recognition [Sang et al., 2005], Friends and Smokers, Students and Professors [Chavira et al., 2006], and `tcc4f`, and • non-binary classic Bayesian networks (`alarm`, `diabetes`, `hailfinder`, `mildew`, `munin1–4`, `pathfinder`, `pigs`, `water`).

Figure 2 shows that `cd05` [Chavira and Darwiche, 2005] and `cd06` [Chavira and Darwiche, 2006] (when run with Ace) are in the lead, while ADDMC [Dudek et al., 2020a] significantly underperforms when combined with any of the previous encodings. Our encoding `cw` significantly improves the performance of ADDMC, making ADDMC + `cw` comparable to Ace + `d02`, c2d + `bklm16`, and Cachet + `sbk05`. Furthermore, Table 2 shows that, while Ace + `cd06` managed to solve the most instances, ADDMC + `cw` was the best-performing algorithm-encoding combination on the largest number of instances. The scatter plot on the left-hand side of Fig. 3 add to this by showing that `cw` is particularly promising on Grid networks and tackles all DQMR instances in less than a second. The scatter plot on the right-hand side of Fig. 3 shows that `cw` is better than `sbk05` [Sang et al., 2005] (i.e., the second-best encoding for ADDMC) on the majority of instances. Seeing how, e.g., DQMR instances are trivial for ADDMC + `cw` but hard for Ace + `cd06`, and vice versa for Mastermind instances, we conclude that the best-performing algorithm-encoding combination depends significantly on (as-of-yet unknown) properties of the Bayesian networks.

Table 3: Asymptotic upper bounds on the numbers of variables and clauses/ADDs for each encoding.

Encoding(s)	Variables	Clauses/ADDs
`bklm16`, `cd05`, `cd06`, `sbk05`	$O(nv^{d+1})$	$O(nv^{d+1})$
`cw`	$O(nv)$	$O(nv^2)$
`d02`	$O(nv^{d+1})$	$O(ndv^{d+1})$

We can explain what makes ADDMC [Dudek et al., 2020a] run significantly faster with `cw` than with any other encoding by considering asymptotic upper bounds on the numbers of variables and ADDs based on the size and structure of the Bayesian network. Let $n = |\mathcal{V}|$ be the number of vertices in the Bayesian network, $d = \max_{X \in \mathcal{V}} |\mathrm{pa}(X)|$ the maximum in-degree (i.e., the number of parents), and $v = \max_{X \in \mathcal{V}} |\operatorname{im} X|$ the maximum number of values per variable. Table 3 shows how `cw` has fewer variables and fewer ADDs than any other encoding. We conjecture that it is primarily the reduced number of variables that makes the ADDMC variable ordering heuristics much more effective. Note that these are upper bounds and most encodings (including `cw`) can be smaller in certain situations (e.g., with binary random variables or when a CPT has repeating probabilities). We equate clauses and ADDs (more specifically, factors of the function ϕ from Algorithm 1) here because ADDMC interprets each clause of any WMC encoding as a multiplicative factor of the ADD that represents the entire WMC instance [Dudek et al., 2020a]. For literal-weight encodings, each weight is also a factor, but that does not affect our asymptotic bounds.

7 CONCLUSIONS AND FUTURE WORK

WMC was originally motivated by an appeal to the success of SAT solvers in efficiently tackling an NP-complete problem [Sang et al., 2005]. ADDMC does not rely on SAT-based algorithmic techniques [Dudek et al., 2020a], and our proposed format diverges even more from the DIMACS CNF format for Boolean formulas. To what extent are SAT-based methods still applicable? The answer depends significantly on the problem domain. For Bayesian networks, the rules describing that each random variable can only be associated with exactly one value were still encoded as clauses. As has been noted previously [Chavira and Darwiche, 2006],

rows in CPTs with probabilities equal to zero or one can be represented as clauses as well. Therefore, our work can be seen as proposing a middle ground between #SAT and probabilistic inference.

While we chose ADDMC [Dudek et al., 2020a] as the WMC algorithm and Bayesian networks as a canonical example of a probabilistic inference task, these are only examples meant to illustrate the broader idea that choosing a more expressive representation of weights can outperform increasing the size of the problem to keep the weights simple. Indeed, in this work, we have provided a new theoretical perspective on the expressive power of WMC and illustrated the empirical benefits of that perspective. Perhaps the same idea could be adapted to other inference problem domains such as probabilistic programs [Fierens et al., 2015, Holtzen et al., 2020] as well as to search-based solvers such as Cachet [Sang et al., 2004] and DPMC —an extension to ADDMC that adds support for computations based on tensors (rather than ADDs) and planning based on tree decompositions Dudek et al. [2020b].

Author Contributions

P. Dilkas conceived the idea, ran the experiments, and wrote the paper. V. Belle supervised the work.

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